

# IRREDUCIBLE INTEGRABLE REPRESENTATIONS OF TOROIDAL LIE ALGEBRAS

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## 1. INTRODUCTION

A toroidal Lie algebra  $\mathcal{T}(\mathfrak{g})$  associated to a finite-dimensional complex simple Lie algebra  $\mathfrak{g}_{fin}$  is the universal central extension of the Lie algebra of polynomial maps from  $(\mathbb{C}^*)^k$  to  $\mathfrak{g}_{fin}$  where  $k$  is a positive integer. From the study of these Lie algebras in [AABGP, RM] it is well known that like the affine Lie algebras, toroidal Lie algebras have a set of real and imaginary roots and one can associate with each real root  $\beta$  of  $\mathcal{T}(\mathfrak{g})$  a Lie subalgebra  $\mathfrak{sl}_2(\beta)$  which is isomorphic to  $\mathfrak{sl}_2(\mathbb{C})$ . A  $\mathcal{T}(\mathfrak{g})$ -module is said to be integrable if it is the direct sum of (possibly infinite) finite-dimensional  $\mathfrak{sl}_2(\beta)$ -modules for all real roots  $\beta$  of  $\mathcal{T}(\mathfrak{g})$ . Integrable representations of toroidal Lie algebras and their quotients have been studied in several papers [CL, L, B, BR, PB, FL, R1, R3, R4, R5]. In [R3] the irreducible integrable  $\mathcal{T}(\mathfrak{g})$ -modules having finite-dimensional weight spaces have been classified. In this paper we give an alternative proof of the results in [R3] and also give a parametrization of the isomorphism classes of irreducible  $\mathcal{T}(\mathfrak{g})$ -modules with finite-dimensional weight spaces.

In contrast to an affine Kac-Moody Lie algebra  $\mathfrak{g}_{aff}$  which has a one-dimensional center, the toroidal Lie algebras have a  $\mathbb{Z}^k$ -graded infinite-dimensional center. This is one of the main sources of difficulty in studying the category  $\mathcal{I}_{fin}^*$  of integrable  $\mathcal{T}(\mathfrak{g})$ -modules with finite-dimensional weight spaces on which the center acts non-trivially. This problem was sorted in [R3] and it was shown that in each graded component of the center there exists at most one element that acts non-trivially on an irreducible  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^*$ . The proof there used results on representations of Heisenberg Lie algebras from [F]. With an understanding of the integral form of the universal enveloping algebra  $\mathcal{U}(\mathfrak{g}_{aff})$  of  $\mathfrak{g}_{aff}$  from [G] and using the action of certain current algebras on a weight vector of an irreducible  $\mathcal{T}(\mathfrak{g})$ -module, in this paper we give an alternate proof of this fact. Besides classifying the simple objects, the methods used here facilitate in the study of the homological properties of the objects in the category  $\mathcal{I}_{fin}^*$  in a way streamlined with [CFK].

The irreducible integrable  $\mathfrak{g}_{aff}$ -modules with finite-dimensional weight spaces had been classified in [C1] and their properties studied in a number of papers including [CP1, CP2, CP3, CG, R3, VV]. Using an approach streamlined with [C1, CFK, CP3] we study the simple objects in the category  $\mathcal{I}_{fin}^*$  and prove that upto twisting by a one-dimensional  $\mathcal{T}(\mathfrak{g})$ -module, every irreducible  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^*$  can be uniquely associated with an orbit for the natural action of  $(\mathbb{C}^*)^{k-1}$  on the set  $\Pi$  of finitely supported functions from  $(\mathbb{C}^*)^{k-1}$  to  $P_{aff}^+$ , the set of dominant integrable weights of the affine Kac-Moody Lie algebra associated to  $\mathfrak{g}_{fin}$ .

The paper is organized as follows. After recalling the structure of the toroidal Lie algebras, in Section 2 we state the results on the representation theory of affine Kac-Moody Lie algebras that play an important role in the classification of the integrable irreducible  $\mathcal{T}(\mathfrak{g})$ -modules. We then prove some preliminary results on integrable  $\mathcal{T}(\mathfrak{g})$ -modules in Section 3 and finally in Section 4 we give an

alternative proof of the classification of irreducible  $\mathcal{T}(\mathfrak{g})$ -modules in  $\mathcal{I}_{fin}^*$ . We also establish a necessary and sufficient condition for two irreducible  $\mathcal{T}(\mathfrak{g})$ -modules in  $\mathcal{I}_{fin}^*$  to be isomorphic.

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## 2. PRELIMINARIES

In this section we fix the notations for the paper and recall the explicit realization of  $k$ -toroidal Lie algebras from [R3, RM].

**2.1.** Throughout the paper  $\mathbf{C}, \mathbf{R}, \mathbf{Z}$  and  $\mathbf{N}$  shall denote the field of complex numbers, real numbers, the set of integers and the set of natural numbers,  $\mathbf{Z}_+$  shall denote the set of non-negative integers and  $\mathbf{C}^*$  shall denote the set of non-zero complex numbers. For a commutative associative algebra  $\mathbf{A}$ , the set of maximal ideals of  $\mathbf{A}$  shall be denoted by  $\max \mathbf{A}$  and for a Lie algebra  $\mathfrak{a}$  the universal enveloping algebra of  $\mathfrak{a}$  shall be denoted by  $\mathcal{U}(\mathfrak{a})$ . For  $k \in \mathbf{N}$ , a  $k$ -tuple of integers  $(m_1, \dots, m_k)$  shall be denoted by  $\mathbf{m}$  and given  $\mathbf{m} \in \mathbf{Z}^k$ ,  $\underline{\mathbf{m}}$  shall denote the  $(k-1)$ -tuple of integers  $(m_2, \dots, m_k)$ .

**2.2.** Let  $\mathfrak{g}_{fin}$  be a finite-dimensional simple Lie algebra of rank  $n$ ,  $\mathfrak{h}_{fin}$  a Cartan subalgebra of  $\mathfrak{g}_{fin}$  and  $R_{fin}$  the set of roots of  $\mathfrak{g}_{fin}$  with respect to  $\mathfrak{h}_{fin}$ . Let  $\{\alpha_i : 1 \leq i \leq n\}$  (respectively  $\{\alpha_i^\vee : 1 \leq i \leq n\}$ ,  $\{\omega_i : 1 \leq i \leq n\}$ ) be a set of simple roots (respectively simple coroots and fundamental weights) of  $\mathfrak{g}_{fin}$  with respect to  $\mathfrak{h}_{fin}$ ,  $R_{fin}^+$  (respectively  $Q_{fin}, P_{fin}$ ) be the corresponding set of positive roots (respectively root lattice and weight lattice) and let  $\theta$  (respectively  $\theta_s$ ) be the highest root (respectively highest short root) of  $R_{fin}^+$  when  $\mathfrak{g}_{fin}$  is simply-laced (respectively nonsimply-laced). Let  $Q_{fin}^+$  and  $P_{fin}^+$  be the  $\mathbf{Z}_+$  span of the simple roots and fundamental weights of  $(\mathfrak{g}_{fin}, \mathfrak{h}_{fin})$ . Let  $\Gamma = P_{fin}/Q_{fin}$ . It is well-known that  $\Gamma$  is a finite group whose elements are of the form  $\omega_i \bmod Q_{fin}$  for  $i = 1, \dots, n$ .

Given  $\alpha \in R_{fin}^\pm$  let  $\mathfrak{g}_{fin}^{\pm\alpha}$  denote the corresponding root space and let  $x_\alpha^\pm \in \mathfrak{g}_{fin}^{\pm\alpha}$  and  $\alpha^\vee \in \mathfrak{h}_{fin}$  be fixed elements such that  $\alpha^\vee = [x_\alpha^+, x_\alpha^-]$  and  $[\alpha^\vee, x_\alpha^\pm] = \pm 2x_\alpha^\pm$ . For  $\lambda \in P_{fin}^+$ , let  $V(\lambda)$  denote the cyclic  $\mathfrak{g}_{fin}$ -module generated by a weight vector  $v_\lambda$  with defining relations:

$$x_\alpha^+ v_\lambda = 0, \forall \alpha \in R_{fin}^+, \quad h.v_\lambda = \lambda(h)v_\lambda, \forall h \in \mathfrak{h}_{fin}, \quad (x_\alpha^-)^{\lambda(\alpha)+1}.v_\lambda = 0, \forall \alpha \in R_{fin}^+.$$

It is well known that  $V(\lambda)$  is an irreducible finite-dimensional  $\mathfrak{g}_{fin}$ -module with highest weight  $\lambda$  and any irreducible finite-dimensional  $\mathfrak{g}_{fin}$ -module is isomorphic to  $V(\lambda)$  for  $\lambda \in P_{fin}^+$ . Given a non-zero weight vector  $u$  in a  $\mathfrak{g}_{fin}$ -module  $V$  we shall denote by  $\text{wt}_{fin}(u)$  the weight of  $u$  with respect to the Cartan subalgebra  $\mathfrak{h}_{fin}$  of  $\mathfrak{g}_{fin}$ .

**2.3.** For a positive integer  $k$ , let  $\mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$  be the Laurent polynomial ring in  $k$  commuting variables  $t_1, \dots, t_k$  and for  $\mathbf{m} = (m_1, \dots, m_k) \in \mathbf{Z}^k$ , let  $t^\mathbf{m}$  (respectively  $t^{\underline{\mathbf{m}}}$ ) denote the element  $t_1^{m_1} \dots t_k^{m_k}$  (respectively  $t_2^{m_2} \dots t_k^{m_k}$ ) in  $\mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$ . Let  $L_k(\mathfrak{g}) = \mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  and  $\mathcal{Z} = \Omega_k/dL_k$  be the space of Kähler differentials spanned by the set of vectors  $\{t^\mathbf{m} K_i, \mathbf{m} \in \mathbf{Z}^k, 1 \leq i \leq k\}$  together with the relation,  $\sum_{i=1}^k r_i t^{\mathbf{r}} K_i = 0$ , for  $\mathbf{r} \in \mathbf{Z}^k$ . Let  $d_i : (L_k(\mathfrak{g}) \oplus \mathcal{Z}) \rightarrow (L_k(\mathfrak{g}) \oplus \mathcal{Z})$ ,  $1 \leq i \leq k$  be the  $k$  derivations on  $L_k(\mathfrak{g} \oplus \mathcal{Z})$  given by:

$$d_i(x \otimes t^\mathbf{m}) = m_i x \otimes t^\mathbf{m}, \quad d_i(t^\mathbf{m} K_j) = m_i t^\mathbf{m} K_j \quad \forall \quad 1 \leq i, j \leq k, \quad (2.1)$$

and let  $D_k$  be the  $\mathbf{C}$  linear span of the derivations  $d_1, d_2, \dots, d_k$ . The  $k$ -toroidal Lie algebra associated to a simple Lie algebra  $\mathfrak{g}_{fin}$  is the vector space  $\mathcal{T}(\mathfrak{g}) = L_k(\mathfrak{g}) \oplus \mathcal{Z} \oplus D_k$  on which the Lie bracket is defined by (2.1) and the following relations:

$$[x \otimes P, y \otimes Q] = [x, y] \otimes PQ + \overline{Q(dP)}(x|y), \quad [x \otimes P, \omega] = 0 \quad [\omega, \omega'] = 0, \quad (2.2)$$

where  $x, y \in \mathfrak{g}$ ,  $P, Q \in \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$ ,  $\omega, \omega' \in \mathcal{Z}$  and  $\overline{Q(dP)}$  is the residue class of  $QdP$  in  $\mathcal{Z}$ . Let  $\mathfrak{h}_{tor} := \mathfrak{h}_{fin} \oplus \mathcal{Z}_0 \oplus D_k$ , where  $\mathcal{Z}_0$  is the subspace of  $\mathcal{Z}$  spanned by the central elements of degree zero. In order to identify  $\mathfrak{h}_{fin}^*$  with a subspace of  $\mathfrak{h}_{tor}^*$ , an element  $\lambda \in \mathfrak{h}_{fin}^*$  is extended to an element of  $\mathfrak{h}_{tor}^*$  by setting  $\lambda(c) = 0 = \lambda(d_i) = 0$ , for all  $c \in \mathcal{Z}_0, 1 \leq i \leq k$ . For  $1 \leq i \leq k$ , define  $\delta_i \in \mathfrak{h}_{tor}^*$  by  $\delta_i|_{\mathfrak{h}_{fin} + \mathcal{Z}_0} = 0$ ,  $\delta_i(d_j) = \delta_{ij}$ , for  $1 \leq j \leq k$ . Given  $\alpha \in R_{fin}$  and  $\mathbf{m} = (m_1, \dots, m_n) \in \mathbf{Z}^k$ , set  $\alpha + \delta_{\mathbf{m}} = \alpha + \sum_{i=1}^k m_i \delta_i$  and let

$$R_{tor}^{re} = \{\alpha + \delta_{\mathbf{m}} : \alpha \in R_{fin}, \mathbf{m} \in \mathbf{Z}^k\}, \quad R_{tor}^{im} = \{\delta_{\mathbf{m}} = \sum_{i=1}^k m_i \delta_i : \mathbf{m} \in \mathbf{Z}^k - \{0\}\}.$$

$R_{tor}^{re}$  and  $R_{tor}^{im}$  are respectively the set of real and imaginary roots of  $\mathcal{T}(\mathfrak{g})$  and  $R_{tor} := R_{tor}^{re} \cup R_{tor}^{im}$  is the set of all roots of  $\mathcal{T}(\mathfrak{g})$  with respect to  $\mathfrak{h}_{tor}$ . The root vector corresponding to a real root  $\alpha + \delta_{\mathbf{m}}$  is of the form  $x_{\alpha} \otimes t^{\mathbf{m}}$  and the root vectors corresponding to an imaginary root  $\delta_{\mathbf{m}}$  are of the form  $h \otimes t^{\mathbf{m}}$  with  $h \in \mathfrak{h}_{fin}$ . Setting  $\alpha_{n+i} := \delta_i - \theta$ , for  $i = 1, \dots, k$ , it can be seen that  $\Delta_{tor} = \{\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}\}$  forms a simple system for  $R_{tor}$ .

Given  $\alpha_{\mathbf{m}} = \alpha + \delta_{\mathbf{m}} \in R_{tor}^{re}$ , with  $\alpha \in R_{fin}^+$  and  $\mathbf{m} \in \mathbf{Z}^k$ , let  $\alpha_{\mathbf{m}}^{\vee} = \alpha^{\vee} + \frac{2}{|\alpha|^2} \sum_i m_i K_i$ . With the given Lie bracket operation on  $\mathcal{T}(\mathfrak{g})$  it is easy to check that the subalgebra of  $\mathcal{T}(\mathfrak{g})$  spanned by  $\{x_{\alpha}^+ \otimes t^{\mathbf{m}}, x_{\alpha}^- \otimes t^{-\mathbf{m}}, \alpha_{\mathbf{m}}^{\vee}\}$  is isomorphic to  $\mathfrak{sl}_2(\mathbf{C})$  and we denote it by  $\mathfrak{sl}_2(\alpha + \delta_{\mathbf{m}})$ .

Since  $\mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$  is a commutative associate algebra with unity over the complex field  $\mathbf{C}$ , we see that the following lemma is a special case of [CFK, Lemma 2.2].

**Lemma.** *Let  $\mathfrak{g}_{fin}$  be a simple Lie algebra. Then any ideal of  $L_k(\mathfrak{g})$ ,  $k \in \mathbb{N}$  is of the form  $\mathfrak{g}_{fin} \otimes S$ , where  $S$  is an ideal of  $\mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$ . Further,*

$$[\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]/S, \mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]/S] = \mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]/S.$$

**2.4.** For  $k = 1$ , the Lie algebra  $\mathcal{T}(\mathfrak{g})$  is called an affine Kac-Moody Lie algebra and we denote it by  $\mathfrak{g}_{aff}$ . Explicitly  $\mathfrak{g}_{aff} = \mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}] \oplus \mathbf{C}K_1 \oplus \mathbf{C}d_1$ . Owing to the natural ordering in  $\mathbf{Z}$ , the set of real and imaginary roots of  $\mathfrak{g}_{aff}$  can be partitioned as follows:

$$R_{aff}^{re\pm} = \{\alpha + m\delta_1 : \alpha \in R_{fin}, m \in \mathbf{Z}_{\pm} \setminus \{0\}\} \cup R_{fin}^{\pm}, \quad R_{aff}^{im\pm} = \{m\delta_1 : m \in \mathbf{Z}_{\pm} \setminus \{0\}\}.$$

The set  $R_{aff}^+ = R_{aff}^{re+} \cup R_{aff}^{im+}$  (respectively  $R_{aff}^- = R_{aff}^{re-} \cup R_{aff}^{im-}$ ) is called the set of positive (respectively negative) roots of  $\mathfrak{g}_{aff}$  and  $R_{aff} = R_{aff}^+ \cup R_{aff}^-$  is the set of roots of  $\mathfrak{g}_{aff}$ . Denoting the root space of  $\mathfrak{g}_{aff}$  corresponding to a root  $\gamma \in R_{aff}$  by  $\mathfrak{g}_{aff}^{\gamma}$ , set  $\mathfrak{n}_{aff}^{\pm} = \bigoplus_{\gamma \in R_{aff}^{\pm}} (\mathfrak{g}_{aff}^{\gamma})$  and  $\mathfrak{h}_{aff} = \mathfrak{h}_{fin} \oplus$

$\mathbf{C}K_1 \oplus \mathbf{C}d_1$ . The set of simple roots  $\Delta_{aff}$  and coroots  $\Delta_{aff}^{\vee}$  of  $\mathfrak{g}_{aff}$  are respectively given by  $\Delta_{aff} = \{\alpha_1, \dots, \alpha_n, \alpha_{n+1} = \delta_1 - \theta\}$ , and  $\Delta_{aff}^{\vee} = \{\alpha_1^{\vee}, \dots, \alpha_n^{\vee}, \alpha_{n+1}^{\vee} = K_1 - \theta^{\vee}\}$ . Let  $Q_{aff}$  (respectively  $Q_{aff}^{\vee}$ ) be the root lattice (respectively coroot lattice) for  $\mathfrak{g}_{aff}$ . Let  $\Lambda_i$  ( $i = 1, \dots, n, n+1$ ) be the fundamental weights of  $\mathfrak{g}_{aff}$ , that is,  $\langle \Lambda_i, \alpha_j^{\vee} \rangle = \delta_{ij}$ , for  $1 \leq j \leq n+1$  and  $\Lambda_i(d_1) = 0$ . Then one has the decomposition

$$\mathfrak{g}_{aff} = \mathfrak{n}_{aff}^- \oplus \mathfrak{h}_{aff} \oplus \mathfrak{n}_{aff}^+, \quad \text{and} \quad \mathfrak{h}_{aff}^* = \mathfrak{h}_{fin}^* \oplus \mathbf{C}\delta_1 \oplus \mathbf{C}\Lambda_{n+1},$$

where  $\delta_1$  is regarded as an element of  $\mathfrak{h}_{aff}^*$  by defining  $\delta_1|_{\mathfrak{h}_{fin}} = 0 = \delta_1(K_1) = 0$ ,  $\delta_1(d_1) = 1$ . Thus an element  $\lambda$  in  $\mathfrak{h}_{aff}^*$  can be uniquely written as

$$\lambda = \lambda(K_1)\Lambda_{n+1} + \lambda|_{\mathfrak{h}_{fin}} + \lambda(d_1)\delta_1,$$

where  $\lambda|_{\mathfrak{h}_{fin}}$  is the restriction of  $\lambda$  to  $\mathfrak{h}_{fin}$ .

Let  $P_{aff} = \sum_{i=1}^{n+1} \mathbf{Z}\Lambda_i + \mathbf{C}\delta_1$ , (respectively  $P_{aff}^+ = \sum_{i=1}^{n+1} \mathbf{Z}_+\Lambda_i + \mathbf{C}\delta_1$ ) be the set of integral weights (respectively dominant integral weights) of  $\mathfrak{g}_{aff}$ . Let  $\succeq$  be the partial order on  $P_{aff}$  defined by  $\lambda \succeq \mu$  if  $\lambda, \mu \in P_{aff}$  are such that  $\lambda - \mu \in \sum_{i=1}^{n+1} \mathbf{Z}_+\alpha_i$ . Given  $\lambda, \mu \in P_{aff}$  we shall write  $\lambda \succ \mu$  whenever  $\lambda \succeq \mu$  but  $\lambda \neq \mu$ .

**2.5.** A  $\mathfrak{g}_{aff}$ -module  $V$  is said to be integrable if it is  $\mathfrak{h}_{aff}$  diagonalizable and the elements  $x_\alpha \otimes t_1^n$ , with  $\alpha \in R_{fin}, n \in \mathbf{Z}$  are locally nilpotent on every  $v \in V$ . The irreducible integrable  $\mathfrak{g}_{aff}$ -modules with finite-dimensional weight spaces were classified in [C1, CP1, CP2]. It was proved that they are either standard modules  $X(\Lambda)$ , restricted duals of standard modules  $X^*(\Lambda)$  or loop modules  $V(\boldsymbol{\lambda}, \mathbf{a}, b)$  which are described as follows.

Given  $\Lambda \in P_{aff}^+$ , a standard module  $X(\Lambda)$  is the unique irreducible  $\mathfrak{g}_{aff}$ -module with highest weight  $\Lambda$  and highest weight vector  $v_\Lambda$  such that  $X(\Lambda) = \mathcal{U}(\mathfrak{g}_{aff}).v_\Lambda$ . Further  $v_\Lambda$  satisfies the relation

$$\mathfrak{n}_{aff}^+.v_\Lambda = 0, \quad h.v_\Lambda = \lambda(h)v_\Lambda, \quad \forall h \in \mathfrak{h}_{aff}, \quad (x_{\alpha_i}^-)^{\Lambda(\alpha_i)+1}.v_\Lambda = 0, \quad \forall 1 \leq i \leq n+1.$$

The restricted dual  $X^*(\Lambda)$  of a standard module  $X(\Lambda)$  is a  $\mathfrak{g}_{aff}$ -module generated by a weight vector  $v_\Lambda^*$  satisfying the relations

$$\mathfrak{n}_{aff}^-.v_\Lambda^* = 0, \quad h.v_\Lambda^* = -\Lambda(h)v_\Lambda^*, \quad \forall h \in \mathfrak{h}_{aff}, \quad (x_{\alpha_i}^+)^{\Lambda(\alpha_i)+1}.v_\Lambda^* = 0, \quad \forall 1 \leq i \leq n+1.$$

Finally, for  $r \in \mathbf{Z}_+$ ,  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in (P_{fin}^+)^r$ ,  $\mathbf{a} = (a_1, \dots, a_r) \in (\mathbf{C}^*)^r$ , and  $b \in \mathbf{C}^*$  the loop module  $V(\boldsymbol{\lambda}, \mathbf{a}, b)$  is the vector space  $V(\lambda_1) \otimes \dots \otimes V(\lambda_r) \otimes \mathbf{C}[t_1^{\pm 1}]$  on which the action of  $\mathfrak{g}_{aff}$  is defined as :

$$\begin{aligned} K_1.(v_1 \otimes \dots \otimes v_r \otimes f) &= 0, \\ x \otimes t_1^s.(v_1 \otimes \dots \otimes v_r \otimes f) &= \sum_{i=1}^r a_i^s v_1 \otimes \dots \otimes x.v_i \otimes \dots \otimes v_r \otimes f t_1^s, \\ d(v_1 \otimes \dots \otimes v_r \otimes t^p) &= s + p v_1 \otimes \dots \otimes v_r \otimes t^p, \end{aligned}$$

where  $v_i \in V(\lambda_i)$  for  $1 \leq i \leq r$ ,  $f \in \mathbf{C}[t_1^{\pm 1}]$ ,  $x \in \mathfrak{g}_{fin}$  and  $p \in \mathbf{Z}$ . Given an integrable  $\mathfrak{g}_{aff}$ -module  $V$ , let  $P_{aff}(V) = \{\eta \in P_{aff} : V_\eta \neq 0\}$ , where  $V_\eta = \{v \in V : hv = \eta(h)v, \text{ for } h \in \mathfrak{h}_{aff}\}$  for  $\eta \in P_{aff}$ . For  $v \in V_\eta$  we write  $\text{wt}_{aff}(v) = \eta$ . With this notation we state the following results. Parts (i) and (ii) of the following proposition have been proved in [CP1, C1] and parts (iii) and (iv) have been proved in [R2, CG].

**Proposition.**

- (i). Let  $r \in \mathbf{N}$ . Given  $\boldsymbol{\lambda} \in (P_{fin}^+)^r$ ,  $\mathbf{a} \in (\mathbf{C}^*)^r$  and  $b \in \mathbf{C}$ , the loop module  $V(\boldsymbol{\lambda}, \mathbf{a}, b)$  is completely reducible as a  $\mathfrak{g}_{aff}$ -module.
- (ii). Let  $V$  be an irreducible representation of  $\mathfrak{g}_{aff}$  having finite-dimensional weight spaces. If  $m \in \mathbf{Z}$  is such that  $K_1.v = mv$ , for all  $v \in V$ , then
  - a. for  $m > 0$ , (respectively  $m < 0$ )  $V$  is isomorphic to  $X(\Lambda)$  (respectively  $X^*(\Lambda)$ ) for some  $\Lambda \in P_{aff}^+$ .
  - b. for  $m = 0$ ,  $V$  is isomorphic to an irreducible component of a loop module  $V(\boldsymbol{\lambda}, \mathbf{a}, b)$  for some  $\boldsymbol{\lambda} \in (P_{fin}^+)^r$ ,  $\mathbf{a} \in (\mathbf{C}^*)^r$ ,  $b \in \mathbf{C}$  and  $r \in \mathbf{Z}$ .

- (iii). If all eigenvalues for the action of the central element  $K_1$  on an integrable  $\mathfrak{g}_{aff}$ -module  $V$ , are non-zero then  $V$  is completely reducible as  $\mathfrak{g}_{aff}$ -module.
- (iv). Let  $\lambda \in P_{aff}^+$  be of the form  $\lambda = \lambda(K_1)\Lambda_{n+1} + \lambda|_{\mathfrak{h}_{fin}} + \lambda(d_1)\delta_1$ . If  $\varpi_\lambda$  is the unique minimal element in  $P_{fin}^+$  such that  $\lambda|_{\mathfrak{h}_{fin}} \equiv \varpi_\lambda \pmod{Q_{fin}}$  then  $\varpi_{\lambda,r} = \lambda(K_1)\Lambda_{n+1} + \varpi_\lambda + r\delta_1 \in P_{aff}(X(\lambda))$  for all  $r \in \mathbf{C}$  such that  $\lambda(d) - r \in \mathbf{Z}_+$ . In particular, if  $\lambda|_{\mathfrak{h}_{fin}} \in P_{fin}^+ \cap Q_{fin}^+$  and  $\lambda|_{\mathfrak{h}_{fin}} \neq 0$ , then  $\lambda(K_1)\Lambda_{n+1} + \beta + r\delta_1 \in P_{aff}(\lambda)$  for all  $r \in \mathbf{C}$  such that  $\lambda(d_1) - r \in \mathbf{Z}_+$ , where  $\beta = \theta$  if  $\mathfrak{g}$  is simply-laced and  $\beta = \theta_s$  otherwise.

**2.6.** Let  $u$  be an indeterminate. For  $\alpha \in R_{fin}$  define a power series  $\mathbf{p}_\alpha(u)$  in  $u$  with coefficients in  $\mathcal{U}(\mathfrak{h}_{fin} \otimes \mathbf{C}[t^{\pm 1}])$  :

$$\mathbf{p}_\alpha(u) = \exp \left( - \sum_{r=1}^{\infty} \frac{\alpha^\vee \otimes t^r}{r} u^r \right).$$

For  $s \in \mathbf{Z}_+$ , let  $p_\alpha^s$  be the coefficient of  $u^s$  in  $\mathbf{p}_\alpha(u)$ . The following was proved in [G, Lemma 7.5].

**Lemma.** Let  $\alpha \in R_{fin}^+$ . Then for  $r \geq 1$  we have

$$(x_\alpha^+ \otimes t)^r (x_\alpha^- \otimes 1)^{r+1} - \sum_{s=0}^r (x_\alpha^- \otimes t^{r-s}) p_\alpha^s \in \mathcal{U}(\mathfrak{g}_{aff})(\mathfrak{n}_{aff}^+), \quad (2.3)$$

$$(x_\alpha^+ \otimes t)^{r+1} (x_\alpha^- \otimes 1)^{r+1} - p_\alpha^{r+1} \in \mathcal{U}(\mathfrak{g}_{aff})(\mathfrak{n}_{aff}^+). \quad (2.4)$$

### 3. INTEGRABLE REPRESENTATIONS OF THE TOROIDAL LIE ALGEBRA

**3.1.** Let  $\mathcal{I}$  be the category whose objects are integrable  $\mathcal{T}(\mathfrak{g})$ -modules and morphisms

$$\mathrm{Hom}_{\mathcal{I}}(V, V') = \mathrm{Hom}_{\mathcal{T}(\mathfrak{g})}(V, V'), \quad V, V' \in \mathcal{I}.$$

For an integrable representation  $\psi : \mathcal{T}(\mathfrak{g}) \rightarrow \mathrm{End}(V)$  of  $\mathcal{T}(\mathfrak{g})$ , let  $V^\psi$  denote the corresponding  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}$ . Given  $\beta = \alpha + \delta_{\mathbf{m}} \in R_{tor}^{re}$  with  $\alpha \in R_{fin}$ , define an operator  $r_\beta^\psi$  on  $V$  as follows:

$$r_\beta^\psi = \exp(\psi(x_\alpha^+ \otimes t^{\mathbf{m}})) \exp(\psi(-x_\alpha^- \otimes t^{-\mathbf{m}})) \exp(\psi(x_\alpha^+ \otimes t^{\mathbf{m}})).$$

Since  $V^\psi$  is an integrable  $\mathcal{T}(\mathfrak{g})$ -module, the operator  $r_\beta^\psi$  is well-defined for all  $\beta \in R_{tor}^{re}$ . Let

$$P(V^\psi) = \{\lambda \in \mathfrak{h}_{tor}^* : V_\lambda^\psi \neq 0\}, \quad \text{where } V_\lambda^\psi = \{v \in V^\psi : h.v = \lambda(h)v, \text{ for } h \in \mathfrak{h}_{tor}\}.$$

Let  $W_{tor}^\psi = \langle r_\beta^\psi : \beta \in R_{tor}^{re} \rangle$ , be the group generated by the operators  $r_\beta^\psi$  for  $\beta \in R_{tor}^{re}$ . By [Kac, Lemma 3.8, §6.5],  $r_\beta^\psi(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta$ , for  $\lambda \in P(V^\psi)$ . Using the representation theory of  $\mathfrak{sl}_2(\mathbf{C})$  the following is standard in  $\mathcal{I}$ .

**Lemma.** Let  $V^\psi$  be a  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}$  and let  $\lambda \in P(V^\psi)$ . Then,

- i.  $\langle \lambda, \alpha^\vee \rangle \in \mathbf{Z}$ , for  $\alpha \in R_{tor}^{re}$ .
- ii.  $w\lambda \in P(V^\psi)$  and  $\dim V_{w\lambda}^\pi = \dim V_{w\lambda}^\psi$  for all  $\lambda \in P(V^\psi)$ ,  $w \in W_{tor}^\psi$ .
- iii. Let  $\alpha \in R_{fin}^+$  and  $\beta = \alpha + m_i \delta_i \in R_{tor}^{re}$ . Given a  $\mathcal{T}(\mathfrak{g})$ -module  $V^\psi$  in  $\mathcal{I}$  and  $\lambda \in P(V^\psi)$ , we have

$$r_\alpha^\psi r_\beta^\psi(\lambda) = \lambda + \frac{2}{(\alpha|\alpha)} (m_i \langle \lambda, K_i \rangle) \alpha - (\langle \lambda, \alpha^\vee \rangle + \frac{2}{(\alpha|\alpha)} m_i \langle \lambda, K_i \rangle) \delta_i.$$

The following is an easy corollary of Lemma 3.1(iii).

**Corollary.** Let  $V^\psi$  be an integrable  $\mathcal{T}(\mathfrak{g})$ -module. Suppose  $\lambda + \sum_{i=1}^k r_i \delta_i \in P(V^\psi)$  satisfies the condition

$$\langle \lambda, \alpha_{n+1}^\vee \rangle = m, \quad \text{and} \quad \langle \lambda, K_j \rangle = 0, \quad \text{for } j = 2, \dots, k. \quad (3.1)$$

Then there exists  $\underline{m} = (m_2, \dots, m_k) \in \mathbf{Z}^{k-1}$  with  $0 \leq m_i < m$  for  $2 \leq i \leq k$  such that

$$\lambda + r_1 \delta_1 + \sum_{i=2}^k m_i \delta_i \in P(V^\psi).$$

*Proof.* Set  $\beta_j = \delta_j + \delta_1 - \theta = \delta_j + \alpha_{n+1}$  for  $j = 2, \dots, k$ . Then by 2.3,

$$\beta_j^\vee = -\theta^\vee + \frac{2}{(\theta|\theta)}(K_j + K_1) = +\alpha_{n+1}^\vee + \frac{2}{(\theta|\theta)}K_j, \quad 2 \leq j \leq k.$$

By definition  $\alpha_{n+1}, \beta_2, \beta_3, \dots, \beta_k \in R_{tor}^e$ , hence  $r_{\alpha_{n+1}}^\psi, r_{\beta_2}^\psi, \dots, r_{\beta_k}^\psi$  are well-defined operators in  $W_{tor}^\psi$ .

Thus given  $\lambda + \sum_{i=1}^k r_i \delta_i \in P(V^\psi)$ , by Lemma 3.1(ii),  $r_{\alpha_{n+1}}^\psi r_{\beta_j}^\psi (\lambda + \sum_{i=1}^k r_i \delta_i) \in P(V^\psi)$  for  $2 \leq j \leq k$ . Now note that

$$\begin{aligned} r_{\alpha_{n+1}}^\psi r_{\beta_j}^\psi (\lambda + \delta_{\mathbf{r}}) &= r_{\alpha_{n+1}}^\psi (\lambda + \delta_{\mathbf{r}} - \langle \lambda + \delta_{\mathbf{r}}, \beta_j^\vee \rangle \beta_j) = r_{\alpha_{n+1}}^\psi (\lambda + \delta_{\mathbf{r}} - \langle \lambda + \delta_{\mathbf{r}}, \frac{2}{(\theta|\theta)}K_j + \alpha_{n+1}^\vee \rangle (\delta_j + \alpha_{n+1})) \\ &= r_{\alpha_{n+1}}^\psi (\lambda + \delta_{\mathbf{r}} - \langle \lambda + \delta_{\mathbf{r}}, \alpha_{n+1}^\vee \rangle \alpha_{n+1}) + \langle \lambda + \delta_{\mathbf{r}}, \frac{2}{(\theta|\theta)}K_j \rangle \alpha_{n+1} - \langle \lambda + \delta_{\mathbf{r}}, \frac{2}{(\theta|\theta)}K_j + \alpha_{n+1}^\vee \rangle \delta_j \\ &= (\lambda + \delta_{\mathbf{r}}) + \langle \lambda + \delta_{\mathbf{r}}, \frac{2}{(\theta|\theta)}K_j \rangle (\alpha_{n+1} - \delta_j) - \langle \lambda + \delta_{\mathbf{r}}, \alpha_{n+1}^\vee \rangle \delta_j. \end{aligned}$$

Since  $\langle \lambda, \alpha_{n+1}^\vee \rangle = m$  and  $\langle \lambda, K_j \rangle = 0$  for  $2 \leq j \leq k$ , it follows that  $\lambda + \delta_{\mathbf{r}} - m\delta_j \in P(V^\psi)$ . Repeating the argument it is easy to see that if  $\lambda + \delta_{\mathbf{r}} \in V^\psi$  satisfies the condition (3.1) and  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_j \geq 0$  for  $2 \leq j \leq k$ , then there exists  $w \in W_{tor}^\psi$  such that  $w(\lambda + \delta_{\mathbf{r}}) = \lambda + r_1 \delta_1 + \sum_{i=2}^k m_i \delta_i$  with  $0 \leq m_i < m$  for  $2 \leq i \leq k$ . If  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_j \leq 0$  for some  $2 \leq i \leq k$  then the same result can be obtained by setting  $\gamma_j = \delta_j - \alpha_{n+1}$  and using the operator  $r_{\gamma_j}$  in place of  $r_{\beta_j}$ .  $\square$

Assuming that we understand that the elements of  $\mathcal{T}(\mathfrak{g})$  act on an object  $V$  of  $\mathcal{I}$  via a Lie algebra homomorphism  $\psi : \mathcal{T}(\mathfrak{g}) \rightarrow \text{End}(V)$ , we shall henceforth drop the superscript  $\psi$  when referring to  $V^\psi$ ,  $W_{tor}^\psi$  etc.

**3.2.** Let  $\mathcal{Z}_0$  be the  $\mathbf{C}$ -span of the zero degree central elements of  $\mathcal{T}(\mathfrak{g})$ . Given a  $\mathcal{T}(\mathfrak{g})$ -module  $V$  and  $\Lambda \in P(V)$ , the restriction  $\Lambda|_{\mathcal{Z}_0}$  is a map from  $\mathcal{Z}_0$  to  $\mathbf{C}$ . In particular when  $V$  is integrable,  $\Lambda|_{\mathcal{Z}_0} \subset \mathbf{Z}$ . For  $\mathbf{n} = (n_1, \dots, n_k) \in \mathbf{Z}^k$  let  $P_{\mathbf{n}}(V) = \{\lambda \in P(V) : \lambda(K_i) = n_i, \text{ for } 1 \leq i \leq k\}$  and let

$$V^{(\mathbf{n})} = \bigoplus_{\lambda \in P_{\mathbf{n}}(V)} V_\lambda.$$

As  $\mathcal{Z}_0$  commutes with  $\mathcal{T}(\mathfrak{g})$ ,  $V^{(\mathbf{n})}$  is a  $\mathcal{T}(\mathfrak{g})$ -module for each  $\mathbf{n} \in \mathbf{Z}^k$  and any  $V \in \text{Ob } \mathcal{I}$  can be decomposed as follows :

$$V = \bigoplus_{\mathbf{n} \in \mathbf{Z}^k} \left( \bigoplus_{\lambda \in P_{\mathbf{n}}(V)} V_\lambda \right) = \bigoplus_{\mathbf{n} \in \mathbf{Z}^k} V^{(\mathbf{n})}.$$

Note that for all  $\lambda \in P_{\mathbf{n}}(V)$ , the restriction  $\lambda|_{\mathcal{Z}_0}$  is a linear functional. Hence by a change of basis for  $\mathcal{Z}_0$  one can always assume that at most one zero degree central element acts non-trivially on  $V$ . Since in a toroidal Lie algebra the choice of basis for  $\mathcal{Z}_0$  is dependent on the choice of generators

of the coordinate ring  $\mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$ , the change of basis matrix  $\mathbf{B} = (b_{ij})_{1 \leq i, j \leq k}$  for  $\mathcal{Z}_0$  induces a homomorphism  $\tilde{\mathbf{B}} : L_k(\mathfrak{g}) \rightarrow L_k(\mathfrak{g})$  given by :

$$\tilde{\mathbf{B}}(x \otimes t^{\mathbf{m}}) = x \otimes t^{\mathbf{m}\mathbf{B}}.$$

Let  $\{\mathbf{e}_i : 1 \leq i \leq k\}$  be the standard basis of  $\mathbb{R}^k$ . Then setting  $a_i = t^{\mathbf{e}_i \mathbf{B}}$  for  $1 \leq i \leq k$ , it is easy to see that  $\tilde{\mathbf{B}} : L_k(\mathfrak{g}) \rightarrow \mathfrak{g}_{fin} \otimes \mathbf{C}[a_1^{\pm 1}, \dots, a_k^{\pm 1}]$  is an isomorphism of multiloop Lie algebras whenever  $\mathbf{B}$  is an integer matrix with determinant  $\pm 1$ . As noted in [R1], the isomorphism  $\tilde{\mathbf{B}}$  can be extended to an isomorphism of toroidal Lie algebras

$$\tilde{\mathbf{b}} : \mathcal{T}(\mathfrak{g}) \rightarrow \mathfrak{g}_{fin} \otimes \mathbf{C}[a_1^{\pm 1}, \dots, a_k^{\pm 1}] \oplus \tilde{\mathcal{Z}} \oplus \tilde{D}_k,$$

where  $\tilde{\mathcal{Z}}$  is the  $\mathbf{C}$  span of  $\{a^{\mathbf{m}} K'_i : \sum_{i=1}^k r_i a^{\mathbf{r}} K'_i = 0, \mathbf{m}, \mathbf{r} \in \mathbf{Z}^k, 1 \leq i \leq k\}$ , with  $K'_i = \sum_{j=1}^k b_{ij} K_j$  for  $1 \leq i \leq k$  and  $\tilde{D}_k$  is the  $\mathbf{C}$ -span of the derivations  $\{\tilde{d}_i = a_i \frac{d}{da_i} : 1 \leq i \leq k\}$ . Defining the  $\mathcal{T}(\mathfrak{g})$  action on  $V^{(\mathbf{n})} \in \text{Ob } \mathcal{I}$  by

$$X.v = \tilde{\mathbf{b}}(X).v, \quad \forall v \in V,$$

it is then easy to see that with respect to the toroidal Lie algebra  $\tilde{\mathbf{b}}(\mathcal{T}(\mathfrak{g}))$ , the module  $V^{(\mathbf{n})}$  is of the form  $V^{(m\mathbf{e}_1)}$  for some  $m \in \mathbf{Z}$ . The following standard result from group theory shows that this is true in general.i.e., upto an isomorphism any indecomposable object in  $\mathcal{I}$  on which the center acts non-trivially is of the form  $V = V^{(m\mathbf{e}_1)}$  for some  $m \in \mathbf{Z}$ .

**Lemma.** *Let  $\{n_1, \dots, n_k\}$  be a set of integers with  $\gcd(n_1, \dots, n_k) \neq 0$ . Then there exists a  $k \times k$  integer matrix  $\mathbf{B} = (b_{ij})$  with determinant  $\pm 1$  such that*

$$\sum_{j=1}^k b_{1j} n_j = \gcd(n_1, \dots, n_k), \quad \text{and} \quad \sum_{j=1}^k b_{ij} n_j = 0, \quad \forall i \geq 2, \quad (3.2)$$

**3.3.** For  $\mathbf{m} \in \mathbf{Z}^k$  let  $\mathcal{I}^{(\mathbf{m})}$  be the full subcategory of  $\mathcal{I}$  whose objects are  $\mathcal{T}(\mathfrak{g})$ -modules on which the zero degree central element  $K_i$ , acts by the integer  $m_i$  for  $1 \leq i \leq k$ . The following is an immediate consequence of Lemma 3.2 .

**Lemma.** *Let  $V$  be an integrable  $\mathcal{T}(\mathfrak{g})$ -module. Then*

$$V = \bigoplus_{\mathbf{m} \in \mathbf{Z}^k} V^{(\mathbf{m})}, \quad \text{where } V^{(\mathbf{m})} = \bigoplus_{\lambda \in P_{\mathbf{m}}(V)} V_{\lambda}.$$

*Upto an isomorphism the component  $V^{(\mathbf{m})}$  of  $V$  is of the form  $V^{(m\mathbf{e}_1)}$  where  $m = \gcd(m_1, \dots, m_k)$ . Further,  $\text{Ext}_{\mathcal{I}}^1(V, U) = 0$  for  $V \in \text{Ob } \mathcal{I}^{(\mathbf{m})}$ ,  $U \in \text{Ob } \mathcal{I}^{(\mathbf{n})}$  with  $\mathbf{m}, \mathbf{n} \in \mathbf{Z}^k$  whenever  $\mathbf{m} \neq \mathbf{n}$ . In particular,*

$$\mathcal{I} = \bigoplus_{\mathbf{m} \in \mathbf{Z}^k} \mathcal{I}^{(\mathbf{m})},$$

*and given  $\mathbf{m} = (m_1, \dots, m_k) \in \mathbf{Z}^k$  the category  $\mathcal{I}^{(\mathbf{m})}$  is equivalent to  $\mathcal{I}^{(m\mathbf{e}_1)}$  where  $m = \gcd(m_1, \dots, m_k)$ .*

Without loss of generality we thus restrict ourselves to the study of the full subcategory  $\mathcal{I}^{(m\mathbf{e}_1)}$  of  $\mathcal{I}$ .

**3.4.** Let  $\mathcal{I}_{fin}$  ( respectively  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$  ) be the full subcategory of  $\mathcal{I}$  ( respectively  $\mathcal{I}^{(m\mathbf{e}_1)}$  ) consisting of integrable  $\mathcal{T}(\mathfrak{g})$ -modules with finite dimensional weight spaces.

**Proposition.** *Let  $V$  be an integrable  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ , where  $m > 0$ . Let  $\mathfrak{n}_{aff}^+$  be the positive root space of the affine Lie subalgebra  $\mathfrak{g}_{aff} = \mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}] \oplus \mathbb{C}K_1 \oplus \mathbb{C}d_1$  of  $\mathcal{T}(\mathfrak{g})$ . Then*

$$V_{aff}^+ = \{v \in V_\lambda : \mathfrak{n}_{aff}^+ \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].v = 0\}$$

*is a non-empty subset of  $V$ .*

*Proof.* For a  $(k-1)$  tuple  $\mathbf{r} = (r_2, r_3, \dots, r_k) \in \mathbf{Z}^{k-1}$ , let

$$V[\mathbf{r}] = \{v \in V_\lambda : d_i v = r_i v, \ 2 \leq i \leq k, \ \lambda \in P(V)\}.$$

Clearly  $V[\mathbf{r}]$  is an integrable  $\mathfrak{g}_{aff}$ -module with finite-dimensional weight spaces. Using 2.3, we can write

$$V[\mathbf{r}] = \bigoplus_{\gamma \in P_{fin}/Q_{fin}} V^\gamma[\mathbf{r}], \quad \text{where } V^\gamma[\mathbf{r}] = \bigoplus_{\varpi_\gamma \equiv \lambda|_{\mathfrak{h}_{fin}} \pmod{Q_{fin}}} (V_\lambda \cap V[\mathbf{r}]).$$

Since  $m > 0$ , by Proposition 2.5(iii),  $V^\gamma[\mathbf{r}]$  is completely reducible as a  $\mathfrak{g}_{aff}$ -module for each  $\gamma \in P_{fin}/Q_{fin}$ . Hence there exists a (possibly infinite)  $\mathfrak{g}_{aff}$ -module filtration  $\dots \subset V_1 \subset V_0 = V^\gamma[\mathbf{r}]$  of  $V^\gamma[\mathbf{r}]$  such that the successive quotients are isomorphic to the  $\mathfrak{g}_{aff}$ -module summands of  $V^\gamma[\mathbf{r}]$ . Without loss of generality we may assume that  $V_j/V_{j+1} \cong_{\mathfrak{g}_{aff}} X(\lambda_j)$  with  $\lambda_j \in P_{aff}^+$ . Setting  $\tilde{\lambda}_j = \lambda_j + \sum_{i=2}^k r_i \delta_i$  we see that there exists  $v_j \in (V_j/V_{j+1})_{\tilde{\lambda}_j}$ , such that  $\mathcal{U}(\mathfrak{g}_{aff})v_j$  is isomorphic as a  $\mathfrak{g}_{aff}$ -module to  $X(\lambda_j)$ . Since  $\lambda_j|_{\mathfrak{h}_{fin}} \equiv \varpi_\gamma \pmod{Q_{fin}}$ , by Proposition 2.5(iv),  $m\Lambda_{n+1} + \varpi_\gamma + \lambda_j(d_1)\delta_1$  is a weight of  $X(\lambda_j)$ . Let  $r_1^j = \min_{1 \leq s \leq j} \lambda_s(d_1)$ . Then by Lemma 2.5(iv)

$$\Lambda_{\gamma, \mathbf{r}}^j = m\Lambda_{n+1} + \varpi_\gamma + r_1^j \delta_1 + \sum_{i=2}^k r_i \delta_i \in P(V_s/V_{s+1})$$

for all  $s \in \mathbf{Z}_+$  such that  $s \leq j$ , whenever  $\lambda_j|_{\mathfrak{h}_{fin}} \notin Q_{fin}^+$  and

$$\Lambda_{0, \mathbf{r}}^j = m\Lambda_{n+1} + \lambda_j(d_1)\delta_1 + \sum_{i=2}^k r_i \delta_i \in P(V_s/V_{s+1})$$

for all  $s \in \mathbf{Z}_+$  such that  $s \leq j$ , whenever  $\lambda_j|_{\mathfrak{h}_{fin}} \in P_{fin}^+ \cap Q_{fin}^+$ . Let  $x_n = \dim V_{\Lambda_{\gamma, \mathbf{r}}^n}$  (respectively  $V_{\Lambda_{0, \mathbf{r}}^n}$ ). From above argument it follows that  $(x_n)_{n \in \mathbb{N}}$  is a sequence of integers such that  $x_j \geq j$  for all  $j \in \mathbb{N}$ . In particular for every  $K \in \mathbb{N}$ ,  $x_n > K$  for all  $n \geq K+1$ . Thus if for some  $\mathbf{r} \in \mathbf{Z}^{k-1}$ ,  $V^\gamma[\mathbf{r}]$  would have an infinite filtration, then the fact that  $\lim_{n \rightarrow \infty} x_n = \infty$  would contradict the finite-dimensionality of the weight spaces of  $V$ . Hence  $V[\mathbf{r}]$  admits a finite  $\mathfrak{g}_{aff}$ -module filtration for every  $\mathbf{r} = (r_2, \dots, r_k) \in \mathbf{Z}^{k-1}$ . Consequently for each  $\gamma \in P_{fin}/Q_{fin}$  and  $\mathbf{r} = (r_2, \dots, r_k) \in \mathbf{Z}^{k-1}$  there exists  $\lambda_{\gamma, \mathbf{r}} \in P_{aff}^+$  such that  $X(\lambda_{\gamma, \mathbf{r}})$  is a summand of  $V^\gamma[\mathbf{r}]$  but  $X(\lambda_{\gamma, \mathbf{r}} + \beta)$  is not a summand of  $V^\gamma[\mathbf{r}]$  for any  $\beta \in R_{aff}^+$ .

Let

$$\mathbf{Z}_m^{k-1} = \{\underline{\mathbf{m}} = (m_2, \dots, m_k) \in \mathbf{Z}^{k-1} : 0 \leq m_i < m, \ 2 \leq i \leq k\}.$$

Since  $\mathbf{Z}_m^{k-1}$  is a finite subset of  $\mathbf{Z}^{k-1}$ , given  $\gamma \in P_{fin}/Q_{fin}$  there exists  $\lambda_\gamma \in P_{aff}^+$  such that  $X(\lambda_\gamma)$  is a summand of  $V^\gamma[\underline{\mathbf{m}}]$  for some  $\underline{\mathbf{m}} \in \mathbf{Z}_m^{k-1}$  and  $\lambda_\gamma \succeq \lambda_{\gamma, \underline{\mathbf{m}}}$  for all  $\underline{\mathbf{m}} \in \mathbf{Z}_m^{k-1}$ . On the other hand by Corollary 3.1, for  $\lambda \in P_{aff}^+$ ,  $\lambda + \delta_{\mathbf{r}} \in P(V^\gamma)$  if and only if  $\lambda + \delta_{\underline{\mathbf{m}}} \in P(V^\gamma)$  for some  $\underline{\mathbf{m}} \in \mathbf{Z}_m^{k-1}$ . Hence it follows that  $\lambda_\gamma \succeq \lambda_{\gamma, \mathbf{r}}$  for all  $\mathbf{r} \in \mathbf{Z}^{k-1}$  implying that there exists a non-zero vector  $v \in V_{\lambda_\gamma + \sum_{i=2}^k m_i \delta_i}$ , with  $\underline{\mathbf{m}} \in \mathbf{Z}_m^{k-1}$  such that  $\mathfrak{n}_{aff}^+ \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].v = 0$ . This completes the proof of the proposition.  $\square$



4. IRREDUCIBLE OBJECTS IN  $\mathcal{I}_{fin}^{(me_1)}, m > 0$ 

In this section we give a parametrization for the irreducible  $\mathcal{T}(\mathfrak{g})$ -modules in  $\mathcal{I}_{fin}^{(me_1)}$ .

**4.1.** The following proposition, which was first proved in [R3], plays an important role in determining the simple objects in  $\mathcal{I}_{fin}^{(me_1)}, m > 0$ . We give here an alternative proof using an understanding of the integral forms in the universal enveloping algebra  $\mathcal{U}(\mathcal{T}(\mathfrak{g}))$  of  $\mathcal{T}(\mathfrak{g})$ .

**Proposition.** *Let  $V$  be an irreducible  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}, m > 0$ . Then  $K_j t^{\mathbf{r}}$  acts trivially on  $V$  for all  $2 \leq j \leq k$  and all monomials  $t^{\mathbf{r}} \in \mathbb{C}[t_1^{\pm}, t_2^{\pm}, \dots, t_k^{\pm}]$ . Further,  $K_1 t^{\mathbf{r}}$  acts trivially on  $V$  whenever  $\mathbf{r} = (r_1, \dots, r_k) \in \mathbb{Z}^k$  is such that  $r_1 \neq 0$ .*

*Proof.* Given  $1 \leq j \leq k$ , and  $\mathbf{r} \in \mathbb{Z}^k - \{\mathbf{0}\}$  let  $\mathcal{N}_{j,\mathbf{r}} = \{u \in V : K_j t^{\mathbf{r}}.u = 0\}$ . Clearly  $\mathcal{N}_{j,\mathbf{r}}$  is a submodule of the irreducible  $\mathcal{T}(\mathfrak{g})$ -module  $V$ . Hence to prove the proposition it suffices to show that  $\mathcal{N}_{j,\mathbf{r}}$  is non-empty for each  $\mathbf{r} \in \mathbb{Z}^k$  whenever  $2 \leq j \leq k$  and  $\mathcal{N}_{1,\mathbf{r}}$  is non-empty whenever  $r_1 \neq 0$ .

We prove the proposition in 3 steps. Let  $\mathbf{r} = (r_1, \dots, r_k) \in \mathbb{Z}^k$ . In step 1 we show that  $\mathcal{N}_{j,\mathbf{r}} \neq \emptyset$  for all  $1 \leq j \leq k$  whenever  $r_1 > 0$ . In step 2 we show that  $\mathcal{N}_{j,\mathbf{r}} \neq \emptyset$  for  $2 \leq j \leq k$  whenever  $r_j \neq 0$  and in step 3 we take care of the remaining cases. That is we show that  $\mathcal{N}_{j,\mathbf{r}} \neq \emptyset$  for  $2 \leq j \leq k$  whenever  $r_j = 0$  and as a consequence  $\mathcal{N}_{1,\mathbf{r}} \neq \emptyset$  whenever  $r_1 < 0$ .

**Step 1.** By Proposition 3.4 the subspace  $V_{aff}^+$  of  $V$  is non-empty and for any  $v \in V_{aff}^+$  we have

$$\begin{aligned} h \otimes t^{\mathbf{p}}.v &= 0, & \forall \mathbf{p} = (p_1, p_2, \dots, p_k) \in \mathbb{Z}^k \text{ with } p_1 > 0, h \in \mathfrak{h}_{fin}, \\ x_{\alpha}^+ \otimes t_i^s.v &= 0 & \forall s \in \mathbb{Z}, \alpha \in R_{fin}^+, 2 \leq i \leq k, \\ x_{\alpha}^- \otimes t^{\mathbf{r}}.v &= 0 & \forall \mathbf{p} = (p_1, p_2, \dots, p_k) \in \mathbb{Z}^k \text{ with } p_1 > 0, \alpha \in R_{fin}^+. \end{aligned} \quad (4.1)$$

Hence given  $\alpha \in R_{fin}^+$  for all  $2 \leq i \leq k$  we have

$$0 = (x_{\alpha}^+ \otimes t_i^s)(x_{\alpha}^- \otimes t^{\mathbf{r}})v = (x_{\alpha}^- \otimes t^{\mathbf{r}})(x_{\alpha}^+ \otimes t_i^s)v + (\alpha^{\vee} \otimes t^{\mathbf{r}} t_i^s + K_i t^{\mathbf{r}} t_i^s).v,$$

whenever  $\mathbf{r} = (r_1, \dots, r_k) \in \mathbb{Z}^k$  is such that  $r_1 > 0$ . Using the relations (4.1) it thus follows that  $K_i t^{\mathbf{s}}.v = 0$  for all  $2 \leq i \leq k$  whenever  $\mathbf{s} = (s_1, \dots, s_k)$  is such that  $s_1 > 0$ , i.e.,  $\mathcal{N}_{i,\mathbf{s}} \neq \emptyset$  for  $2 \leq i \leq k$  whenever  $s_1 > 0$ . As  $\sum_{j=1}^k s_j K_j t^{\mathbf{s}} = 0$  it follows that  $\mathcal{N}_{1,\mathbf{s}} \neq \emptyset$  whenever  $s_1 > 0$ .

**Step 2.** For a contradiction assume that there exists  $2 \leq j \leq k$  and  $\mathbf{r} \in \mathbb{Z}^k$  with  $r_1 = 0$  such that  $\mathcal{N}_{j,\mathbf{r}} = \emptyset$ . This implies that  $K_j t^{\mathbf{r}}.v \neq 0$  for all  $v \in V_{aff}^+$  and by leml2rep(ii), given  $w$  in  $W_{tor}$ ,  $w.(K_j t^{\mathbf{r}}.v) \neq 0$  whenever  $K_j t^{\mathbf{r}}.v \neq 0$ . Using Corollary 3.1 assume that  $v_0$  is a fixed vector in  $V_{aff}^+$  with  $K_j t^{\mathbf{r}}.v_0 \neq 0$  and weight of  $v_0$  is  $\lambda = \lambda|_{\mathfrak{h}_{aff}} + \sum_{i=2}^k m_i \delta_i$ , with  $r_i - m \leq m_i < r_i$  for  $2 \leq i \leq k$  and  $\lambda|_{\mathfrak{h}_{aff}} \in P_{aff}^+$ .

Let  $\mathcal{H}$  be the  $\mathbb{Z}^{k-1}$ -graded Lie subalgebra of  $\mathfrak{h}_{fin} \otimes \mathbb{C}[t_1^{\pm}, \dots, t_k^{\pm}] \oplus \mathcal{Z} \oplus D_k$  generated by  $\mathfrak{h}_{aff} \otimes \mathbb{C}[t_2, \dots, t_k]$ . Let  $\mathcal{V}^{\lambda} = \mathcal{U}(\mathcal{H}).v_0$  be the  $\mathcal{H}$ -module generated by  $v_0$ . Clearly  $\mathcal{H}$  is a solvable Lie algebra and  $\mathcal{V}^{\lambda}$  is a  $\mathcal{H}$ -submodule of  $V_{aff}^+$ . Fixing

$$\eta_j = \alpha_{n+1} - \delta_j, \quad \text{and} \quad w_j = r_{\alpha_{n+1}} r_{\eta_j}, \quad \text{for } 2 \leq j \leq k,$$

and setting  $\mathbf{w}^{\underline{\ell}} = w_k^{l_k} \cdots w_3^{l_3} w_2^{l_2}$  for  $\underline{\ell} = (l_2, \dots, l_k) \in \mathbb{Z}_+^{k-1} - \{\mathbf{0}\}$ , we see that

$$\mathcal{V}^{\lambda, \underline{\ell}} := \mathcal{U}(\mathcal{H}).\mathbf{w}^{\underline{\ell}}(v_0)$$

is a proper submodule of  $\mathcal{V}^\lambda$  and the corresponding quotient space  $\mathcal{V}^\lambda/\mathcal{V}^{\lambda,\ell}$  is a non-zero finite-dimensional module for the solvable Lie algebra  $\mathcal{H}$ . Hence by Lie's theorem there exists  $\bar{u}_0 \in \mathcal{V}^\lambda/\mathcal{V}^{\lambda,\ell}$  and a function  $\phi : \mathcal{H} \rightarrow \mathbf{C}$  such that

$$h \otimes t_2^{p_2} \cdots t_k^{p_k} \cdot \bar{u}_0 = \phi(h, \underline{\mathbf{p}}) \bar{u}_0, \quad \forall h \in \mathfrak{h}_{aff}, \underline{\mathbf{p}} = (p_2, \dots, p_k) \in \mathbf{Z}_+^{k-1}.$$

In particular,

$$\begin{aligned} h \otimes t_j^{p_j} \cdot \bar{u}_0 &= \phi(h, (0, \dots, 0, p_j, 0, \dots, 0)) \bar{u}_0, \\ h' \otimes t_2^{p_2} \cdots t_{j-1}^{p_{j-1}} t_{j+1}^{p_{j+1}} \cdots t_k^{p_k} \cdot \bar{u}_0 &= \phi(h, (p_2, \dots, p_{j-1}, 0, p_{j+1}, \dots, p_k)) \bar{u}_0, \end{aligned}$$

for  $h, h' \in \mathfrak{h}_{aff}, p_2, \dots, p_k \in \mathbf{Z}_+$ . Hence using the fact that the Killing form  $(\cdot | \cdot)$  is non-degenerate on  $\mathfrak{h}_{fin}$  and

$$[h \otimes t_j^{p_j}, h' \otimes t_2^{p_2} \cdots t_{j-1}^{p_{j-1}} t_{j+1}^{p_{j+1}} \cdots t_k^{p_k}] \cdot \bar{u}_0 = (h|h') p_j K_j t^{\underline{\mathbf{p}}} \cdot \bar{u}_0$$

we get that  $K_j t^{\underline{\mathbf{p}}} \cdot \bar{u}_0 = 0$  for all  $\underline{\mathbf{p}} \in \mathbf{Z}_+^{k-1}$  with  $p_j \neq 0$ . This implies that graded central elements  $K_j t^{\underline{\mathbf{p}}}$  act trivially on  $\bar{u}_0$  or equivalently there exists a non-zero weight vector  $u_0 \in \mathcal{V}^\lambda$  such that for all  $\underline{\mathbf{p}} = (p_2, \dots, p_k) \in \mathbf{Z}_+^{k-1}$  with  $p_j \neq 0$  and  $2 \leq j \leq k$ ,

$$K_j t^{\underline{\mathbf{p}}} \cdot u_0 \in \mathcal{V}^{\lambda, \ell}.$$

By construction, however, for every weight vector  $u \in \mathcal{V}^{\lambda, \ell}$ ,

$$d_j \cdot u \geq m_j + ml_j, \quad \text{for all } j \geq 2.$$

Hence it follows that  $K_j t^{\underline{\mathbf{p}}} \cdot u_0 = 0$  for all  $\underline{\mathbf{p}} \in \mathbf{Z}_+^{k-1} - \{\mathbf{0}\}$  with  $0 \leq p_s < m_s + ml_s$  and  $2 \leq s \leq k$  whenever  $p_j \neq 0$ . Choosing  $\underline{\mathbf{p}} \in \mathbf{Z}_+^{k-1}$  appropriately we thus see that  $K_j t^{\underline{\mathbf{p}}} \cdot u_0 = 0$  for  $2 \leq j \leq k$  which is a contradiction to our initial assumption in the case when  $\mathbf{r} = (r_1, \dots, r_k) \in \mathbf{Z}^k$  is such that  $r_1 = 0$ . This shows that  $\mathcal{N}_{j, \mathbf{r}} \neq 0$  for  $2 \leq j \leq k$  and  $\underline{\mathbf{r}} \in \mathbf{Z}^{k-1}$  whenever  $r_1 = 0$  and  $r_j \neq 0$ .

Consider now the case when  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_1 < 0$ . Let  $V_{aff}^- = \{u \in V : \mathbf{n}_{aff}^- \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \cdot u = 0\}$ . If  $V_{aff}^-$  is a non-empty subset of  $V$ , then by description,

$$h \otimes t^{\mathbf{r}} \cdot u = 0, \quad x_\alpha^- \otimes t_i^s \cdot u = 0, \quad x_\alpha^+ \otimes t^{\mathbf{r}} \cdot u = 0$$

for all  $u \in V_{aff}^-$  and  $h \in \mathfrak{h}_{fin}$ ,  $\alpha \in R_{fin}^+$ ,  $2 \leq i \leq k$ ,  $s \in \mathbf{Z}$ , whenever  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_1 < 0$ . Hence  $\mathcal{U}(\mathfrak{h}_{fin} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]) \cdot u \subset V_{aff}^-$  for all  $u \in V_{aff}^-$  and following the same arguments as in step 1 we can conclude that  $\mathcal{N}_{j, \mathbf{r}} \neq 0$  for  $1 \leq j \leq k$  whenever  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_1 < 0$ . On the contrary suppose that  $V_{aff}^- = \emptyset$  and for every  $u \in V$  there exists  $t^{\mathbf{r}} \in \mathbf{C}[t_1^{-1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  such that  $h \otimes t^{\mathbf{r}} \cdot u \neq 0$  for some  $h \in \mathfrak{h}_{fin}$ . Then given a non-zero weight vector  $v_0 \in V_{aff}^+$ , there exists  $t^{\underline{\mathbf{p}}} \in \mathbf{C}[t_1^{-1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  such that  $-p_1$  is the least positive integer for which  $h \otimes t^{\underline{\mathbf{p}}} \cdot v_0 \neq 0$ . Let

$$\mathcal{Y}^\lambda = \mathcal{U}(\mathcal{H}^a) \cdot v_0 \quad \text{and} \quad \mathcal{Y}^{\lambda, \ell} = \mathcal{U}(\mathcal{H}^a) \cdot \mathbf{s}^\ell (h \otimes t^{\underline{\mathbf{p}}} \cdot v_0),$$

where  $\mathcal{H}^a$  is the solvable Lie subalgebra of  $\mathfrak{h}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathbf{Z} \oplus D_k$  generated by  $\mathfrak{h} \otimes \mathbf{C}[t_1^{-1}, t_2^{a_2}, \dots, t_k^{a_k}]$ , where  $a_j = (\text{sgn } p_j)1$  for  $j \geq 2$  and  $\mathbf{s}^\ell = s_2^{l_2} \cdots s_k^{l_k}$  is an element of the Weyl group  $W_{tor}$  of  $\mathcal{T}(\mathfrak{g})$  for  $\underline{\ell} = (l_2, \dots, l_k) \in \mathbf{Z}_+^{k-1}$  with  $s_j = r_{\alpha_{n+1}} r_{(\alpha_{n+1} - \delta_j)}$  when  $a_j > 0$  and  $s_j = r_{\alpha_{n+1}} r_{(\alpha_{n+1} + \delta_j)}$  when  $a_j < 0$ . Clearly  $\mathcal{Y}^{\lambda, \ell}$  is a  $\mathcal{H}^a$ -submodule of  $\mathcal{Y}^\lambda$  and the corresponding quotient  $\mathcal{Y}^\lambda/\mathcal{Y}^{\lambda, \ell}$  is a finite-dimensional module for  $\mathcal{H}^a$ . Thus by Lie's theorem there exists  $\bar{\omega}_0 \in \mathcal{Y}^\lambda/\mathcal{Y}^{\lambda, \ell}$  and an algebra homomorphism  $\psi : \mathcal{H}^a \rightarrow \mathbf{C}$  such that

$$h \otimes t^{\mathbf{r}} \cdot \bar{\omega}_0 = \psi(h, \mathbf{r}) \bar{\omega}_0, \quad \forall h \otimes t^{\mathbf{r}} \in \mathfrak{h}_{fin} \otimes \mathbf{C}[t_1^{-1}, t_2^{a_2}, \dots, t_k^{a_k}].$$

By the same arguments as above this implies that  $K_j t^{\mathbf{r}} \cdot \bar{\omega}_0 = 0$  for all  $t^{\mathbf{r}} \in \mathbf{C}[t_1^{-1}, t_2^{a_2}, \dots, t_k^{a_k}]$  and  $2 \leq j \leq k$  whenever  $r_j \neq 0$ . Choosing  $\underline{\mathbf{p}} \in \mathbf{Z}_+^{k-1}$  appropriately and repeating the same arguments

as above we conclude that  $K_j t^{\mathbf{r}}$  acts trivially on  $V$  for all  $t^{\mathbf{r}} \in \mathbf{C}[t_1^{-1}, t_2^{a_2}, \dots, t_k^{a_k}]$  whenever  $r_j \neq 0$ . It is now easy to see that modifying the first part of step 2 of the proof appropriately and using the fact that  $\sum_{j=1}^k r_j K_j t^{\mathbf{r}} = 0$ , one can show that  $K_j t^{\mathbf{r}}$  acts trivially on  $V$  for  $1 \leq j \leq k$  whenever  $\mathbf{r} = (r_1, \dots, r_k) \in \mathbf{Z}^k$  is such that  $r_1 < 0$  and  $r_j \neq 0$ .

**Step 3.** To complete the proof of the proposition it remains to show that  $\mathcal{N}_{j,\mathbf{r}} \neq 0$  for  $2 \leq j \leq k$  with  $\mathbf{r} \in \mathbf{Z}^k$  such that  $r_j = 0$  and  $r_1 \leq 0$  and consequently  $\mathcal{N}_{1,\mathbf{r}} \neq 0$  whenever  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_1 < 0$ . By Proposition 3.4,  $V_{aff}^+$  is non-empty and by step 1 of the proof the subspace  $\mathcal{U}(\mathfrak{h} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathcal{Z}).v$  of  $V$  is contained in  $V_{aff}^+$  for all  $v \in V_{aff}^+$ . Thus if  $\lambda \in P_{aff}^+$  is such that  $h.v = \lambda(h)v$ , for  $v \in V_{aff}^+$  and  $h \in \mathfrak{h}_{fin}$ , then for all  $u \in \mathcal{U}(\mathfrak{h} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathcal{Z}).v$ ,  $u \in V_{aff}^+$  and

$$h.u = \lambda(h)u, \quad \forall h \in \mathfrak{h}_{fin}.$$

Since  $V$  is integrable, using the representation theory of  $\mathfrak{sl}_2(\mathbf{C})$  it follows that for any  $\mathbf{r} \in \mathbf{Z}^k$ , and  $\alpha \in R_{fin}^+$ ,

$$(x_{\alpha}^{-} \otimes t^{\mathbf{r}})^{\lambda(\alpha^{\vee})+1}.u = 0, \quad \forall u \in V_{aff}^+.$$

Given  $\mathbf{r} = (r_1, \dots, r_k) \in \mathbf{Z}^k$  set  $\bar{\mathbf{r}}_j = (r_1, \dots, r_{j-1}, 0, r_{j+1}, \dots, r_k)$  for  $1 \leq j \leq k$ . By definition

$$[x_{\alpha}^{+} \otimes t_i^{-r_i} t^{\bar{\mathbf{r}}_i}, x_{\alpha}^{-} \otimes t_i^{r_i}] = -r_i K_i t^{\bar{\mathbf{r}}_i} + \sum_{2 \leq j \leq k, j \neq i, 1} r_j K_j t^{\bar{\mathbf{r}}_i}.$$

But by step 2  $K_j t^{\mathbf{r}}.v = 0$  for all  $v \in V$  whenever  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_j \neq 0$ . Hence,

$$[x_{\alpha}^{+} \otimes t_i^{-r_i} t^{\bar{\mathbf{r}}_i}, x_{\alpha}^{-} \otimes t_i^{r_i}].u = -r_i K_i t^{\bar{\mathbf{r}}_i}.u, \quad \text{whenever } u \in V_{aff}^+.$$

For  $1 \leq i \leq k$ , set

$$D_{\alpha,i}^{\bar{\mathbf{r}}_i}(u) = \exp\left(-\sum_{r=1}^{\infty} \frac{\alpha^{\vee} \otimes (t^{\bar{\mathbf{r}}_i})^s + K_i (t^{\bar{\mathbf{r}}_i})^s}{s} u^s\right) = \sum_{\ell=1}^{\infty} D_{\alpha,i}^{\bar{\mathbf{r}}_i}(\ell) u^{\ell}.$$

If  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_1 = 0$  and  $r_i \neq 0$  for some  $i \geq 2$ , then using the integrability condition on a vector  $v \in V_{aff}^+ \cap V_{\lambda}$  and the fact that  $K_j t^{\mathbf{r}}$ ,  $2 \leq j \leq k$ , acts trivially on  $V$  whenever  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_j \neq 0$  we see that,

$$\frac{(x_{\alpha}^{+} \otimes t_i^{-r_i} t^{\bar{\mathbf{r}}_i})^s}{s!} \frac{(x_{\alpha}^{-} \otimes t_i^{r_i})^{s+1}}{(s+1)!}.v = \sum_{\ell=0}^s (x_{\alpha}^{-} \otimes t_i^{r_i} (t^{\bar{\mathbf{r}}_i})^{s-\ell}) D_{\alpha,i}^{\bar{\mathbf{r}}_i}(\ell).v = 0, \quad \text{whenever } s \geq \lambda(\alpha^{\vee}).$$

Applying  $x_{\alpha}^{+}$  to the above equation we see that for  $v \in V_{aff}^+$

$$\alpha^{\vee} \otimes t_i^{r_i} (t^{\bar{\mathbf{r}}_i})^{\lambda(\alpha^{\vee})}.v = -\left(\sum_{\ell=1}^{\lambda(\alpha^{\vee})} (\alpha^{\vee} \otimes t_i^{r_i} (t^{\bar{\mathbf{r}}_i})^{s-\ell}) D_{\alpha,i}^{\bar{\mathbf{r}}_i}(\ell)\right).v$$

By Step 2 it follows that for all  $v \in V_{aff}^+ \cap V_{\lambda}$ ,

$$[h \otimes (t_j^{-r_j})^{(\lambda(\alpha^{\vee})-1)}, \alpha^{\vee} \otimes t_i^{r_i} (t^{\bar{\mathbf{r}}_i})^{\lambda(\alpha^{\vee})}].v = -r_j (h|\alpha^{\vee})(\lambda(\alpha^{\vee})-1) K_j t_j^{r_j} t_i^{r_i} \left(\prod_{s \neq 1, i, j} t_s^{r_s}\right)^{\lambda(\alpha^{\vee})}.v = 0.$$

Hence

$$\sum_{\ell=1}^{\lambda(\alpha^{\vee})} [h \otimes (t_j^{-r_j})^{(\lambda(\alpha^{\vee})-1)}, (\alpha^{\vee} \otimes t_i^{r_i} (t^{\bar{\mathbf{r}}_i})^{(\lambda(\alpha^{\vee})-\ell)}) D_{\alpha,i}^{\bar{\mathbf{r}}_i}(\ell)].v = 0, \quad \forall v \in V_{aff}^+.$$

Again using the fact that  $K_j t^{\mathbf{r}}$  acts trivially on  $V$  whenever  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_j \neq 0$  we deduce from the above equation that for all  $v \in V_{aff}^+$ ,

$$\begin{aligned} r_j(\lambda(\alpha^\vee) - 1)(h|\alpha^\vee)[K_j(t_i^{r_i}(\prod_{s \neq 1, i, j} t_s^{r_s})^{\lambda(\alpha^\vee)-1})(\alpha^\vee \otimes t^{\bar{\mathbf{r}}_i} + K_i t^{\bar{\mathbf{r}}_i}) + \\ - (\alpha^\vee \otimes t^{\mathbf{r}})K_j(\prod_{s \neq 1, i, j} t_s^{r_s})^{\lambda(\alpha^\vee)-1} + \\ 2(\alpha^\vee \otimes t_i^{r_i})(\alpha^\vee \otimes (t^{\bar{\mathbf{r}}_i}) + K_i(t^{\bar{\mathbf{r}}_i}))K_j(\prod_{s \neq 1, i, j} t_s^{r_s})^{\lambda(\alpha^\vee)-1}]v = 0. \end{aligned}$$

Since the bilinear form  $(\cdot|\cdot)$  is non-degenerate on  $\mathfrak{h}_{fin}$ , choosing  $h \in \mathfrak{h}_{fin}$  appropriately we see that for any  $v \in V_{aff}^+$ ,

$$\begin{aligned} (\alpha^\vee \otimes t^{\mathbf{r}})K_j(\prod_{s \neq 1, i, j} t_s^{r_s})^{\lambda(\alpha^\vee)-1}v = K_j(t_i^{r_i}(\prod_{s \neq 1, i, j} t_s^{r_s})^{\lambda(\alpha^\vee)-1})(\alpha^\vee \otimes t^{\bar{\mathbf{r}}_i} + K_i t^{\bar{\mathbf{r}}_i}) + \\ 2(\alpha^\vee \otimes t_i^{r_i})(\alpha^\vee \otimes (t^{\bar{\mathbf{r}}_i}) + K_i(t^{\bar{\mathbf{r}}_i}))K_j(\prod_{s \neq 1, i, j} t_s^{r_s})^{\lambda(\alpha^\vee)-1}v. \end{aligned}$$

Now applying  $h \otimes t_i^{-r_i}$  to the above equation and using first part of the proof and the fact that  $K_i$  acts trivially on  $V$  for  $i \neq 1$  we see that there exists  $v_0 \in V_{aff}^+$  such that  $K_i t^{\bar{\mathbf{r}}_i}.v_0 = 0$ , for  $i \geq 2$ . Since  $\mathbf{r} \in \mathbf{Z}^k$  is arbitrary we see that  $K_i t^{\mathbf{r}}$  acts trivially on  $V$  for all  $\mathbf{r} \in \mathbf{Z}^k$  such that  $r_1 = 0 = r_i$ .

Finally we consider the case when  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_1 < 0$  and  $r_i \neq 0$  for some  $2 \leq i \leq k$ . Using the integrability condition as above we see that for  $v \in V_{aff}^+ \cap V_\lambda$ ,

$$\frac{(x_\alpha^+ \otimes t_1^{-r_1} t^{\bar{\mathbf{r}}_1})^s}{s!} \frac{(x_\alpha^- \otimes t_1^{r_1})^{s+1}}{(s+1)!} v = \sum_{\ell=0}^s (x_\alpha^- \otimes t_1^{-r_1} (t^{\bar{\mathbf{r}}_1})^{s-\ell}) D_{\alpha,1}^{\bar{\mathbf{r}}_1}(\ell).v = 0, \quad \text{whenever } s \geq \lambda(\alpha^\vee).$$

Applying  $\text{ad}(h \otimes (t_i^{-r_i})^{\lambda(\alpha^\vee)-1}) \text{ad } x_\alpha^+$  to the above equation and using the first part of the proof we get that for  $v \in V_{aff}^+ \cap V_\lambda$ ,

$$\begin{aligned} 0 = K_i t_1^{r_1} t_i^{r_i} (\prod_{s \neq 1, i} t_s^{r_s})^{\lambda(\alpha^\vee)}.v = K_i t_1^{r_1} (\prod_{s \neq 1, i} t_s^{r_s})^{\lambda(\alpha^\vee)-1} (\alpha^\vee \otimes t_1^{\bar{\mathbf{r}}_1} + K_1 t^{\bar{\mathbf{r}}_1}).v \\ - (\alpha^\vee \otimes t^{\mathbf{r}})(K_i (\prod_{s \neq 1, i} t_s^{r_s})^{\lambda(\alpha^\vee)-1}).v \\ + 2(\alpha^\vee \otimes t_1^{r_1})(\alpha^\vee \otimes t^{\bar{\mathbf{r}}_1} + K_1 t^{\bar{\mathbf{r}}_1})K_i (\prod_{s \neq i, i} t_s^{r_s})^{\lambda(\alpha^\vee)-1}.v, \end{aligned}$$

which implies that for all  $v \in V_{aff}^+ \cap V_\lambda$ ,

$$\begin{aligned} (\alpha^\vee \otimes t^{\mathbf{r}})(K_i (\prod_{s \neq 1, i} t_s^{r_s})^{\lambda(\alpha^\vee)-1}).v = K_i t_1^{r_1} (\prod_{s \neq 1, i} t_s^{r_s})^{\lambda(\alpha^\vee)-1} (\alpha^\vee \otimes t_1^{\bar{\mathbf{r}}_1} + K_1 t^{\bar{\mathbf{r}}_1}).v \\ + 2(\alpha^\vee \otimes t_1^{r_1})(\alpha^\vee \otimes t^{\bar{\mathbf{r}}_1} + K_1 t^{\bar{\mathbf{r}}_1})K_i (\prod_{s \neq i, i} t_s^{r_s})^{\lambda(\alpha^\vee)-1}.v. \end{aligned}$$

Applying  $\text{ad}(h \otimes t_i^{-r_i})$  to the above equation and using the fact that for  $2 \leq i \leq k$ ,  $K_i t^{\mathbf{r}}$  acts trivially on  $V$  whenever  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_1 = 0 = r_i$  or  $r_i \neq 0$  and  $r_1 < 0$ , we see that  $\mathcal{N}_{i, \bar{\mathbf{r}}_i} \cap (V_{aff}^+ \cap V_\lambda) \neq 0$ .

Since for any  $\mathbf{r} \in \mathbf{Z}^k$ ,  $\sum_{j=1}^k r_j K_j t^{\mathbf{r}} = 0$  it follows that  $\mathcal{N}_{1, \mathbf{r}} \neq 0$  whenever  $r_1 < 0$ . This completes the proof of the proposition.  $\square$

It follows from Proposition 4.1 that an irreducible  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$  is in fact a module for the Lie algebra  $\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathcal{Z}_1 \oplus D_k$ , where  $\mathcal{Z}_1$  is the subspace of  $\mathcal{Z}$  spanned by the central elements  $K_1 t^{\mathbf{r}}$ , with  $\mathbf{r} = (r_1, \dots, r_k) \in \mathbf{Z}^k$  such that  $r_1 = 0$ .

**4.2.** Let  $V$  be an irreducible  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ . By Proposition 3.4 there exists a non-zero weight vector  $v_0 \in V_{aff}^+$  such that  $V = \mathcal{U}(\mathcal{T}(\mathfrak{g})).v_0$  and

$$\mathfrak{n}_{aff}^+ \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].v_0 = 0, \quad h.v_0 = \Lambda(h).v_0 \quad \forall h \in \mathfrak{h}_{aff}, \quad (x_i^-)^{\Lambda(\alpha_i^\vee)+1}.v_0 = 0, \quad \text{for } i = 1, \dots, n+1. \quad (4.2)$$

Clearly  $V_{aff}^+ \subseteq \mathcal{U}(\mathfrak{h}_{aff} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]).v_0$ . For  $\alpha \in R_{fin}$  and  $r \in \mathbf{Z}$  the triple  $\{x_\alpha^\pm \otimes t_1^{\pm r}, \alpha^\vee - 2r/(\alpha|\alpha)K_1\}$  is isomorphic to a copy of  $\mathfrak{sl}_2(\mathbf{C}) = \mathbf{C}x^+ \oplus \mathbf{C}x^- \oplus \mathbf{C}h$ . Hence given a monomial

$a \in \mathbf{C}[t_2^{\pm}, \dots, t_k^{\pm}]$ , the map  $\phi_{\alpha,r}^a : \mathfrak{sl}_2(\mathbf{C}) \otimes \mathbf{C}[t^{\pm 1}] \rightarrow \mathfrak{sl}_2(\mathbf{C}) \otimes \mathbf{C}[a^{\pm 1}]$  given by

$$x^{\pm} \otimes t^s \mapsto (x_{\alpha}^{\pm} \otimes t^{\pm r}) \otimes a^s, \quad h \otimes t^s \mapsto (\alpha^{\vee} - 2r/(\alpha|\alpha)K_1) \otimes a^s, \quad \forall s \in \mathbf{Z},$$

defines a Lie algebra homomorphism. For  $\beta = \alpha + r\delta_1 \in R_{aff}^{e+}$ , let  $p_{\beta,a}^s = \phi_{\alpha,r}^a(p_{\alpha}^s)$ . Using the homomorphism  $\phi_{\alpha,r}^a$  and Lemma 2.6 it is easy to see that for  $v_0 \in V_{aff}^+$  we have,

$$(x_{\beta}^+ \otimes a)^s (x_{\beta}^- \otimes 1)^{s+1} \cdot v_0 = \sum_{l=0}^s (x_{\beta}^- \otimes a^{s-l}) p_{\beta,a}^l \cdot v_0, \quad (4.3)$$

$$(x_{\beta}^+ \otimes a)^{s+1} (x_{\beta}^- \otimes a)^{s+1} \cdot v_0 = p_{\beta,a}^{s+1} \cdot v_0. \quad (4.4)$$

Note that  $p_{\beta,a}^l \in \mathcal{U}(\mathfrak{h}_{aff} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}])$ , hence applying  $x_{\beta}^+$  to equation (4.3) we get

$$x_{\beta}^+ (x_{\beta}^+ \otimes a)^s (x_{\beta}^- \otimes 1)^{s+1} \cdot v_0 = \sum_{l=0}^s (\beta^{\vee} \otimes a^{s-l}) p_{\beta,a}^l \cdot v_0.$$

Given  $\Lambda \in P_{aff}^+$  and  $v_0 \in V_{\Lambda} \cap V_{aff}^+$ , we thus conclude from (4.2), (4.3) and (4.4), that for all  $\beta \in R_{aff}^{e+}$  and  $s \geq \Lambda(\beta^{\vee}) + 1$ ,

$$p_{\beta,a}^s \cdot v_0 = 0, \quad \beta^{\vee} \otimes a^s \cdot v_0 + \left( \sum_{l=1}^s (\beta^{\vee} \otimes a^{s-l}) p_{\beta,a}^l \right) \cdot v_0 = 0. \quad (4.5)$$

For a fixed  $\Lambda \in P_{aff}^+$ , let  $I_{\Lambda}$  be the ideal of  $\mathcal{U}(\mathfrak{h}_{aff} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}])$  generated by the elements:

$$h - \Lambda(h), \quad \forall h \in \mathfrak{h}_{aff}, \quad K_j t^{\mathbf{m}} \quad 2 \leq j \leq k, \text{ and all monomials } t^{\mathbf{m}} \text{ in } \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$$

$$p_{\alpha_i,a}^s, \quad \text{for } |s| > \Lambda(\alpha_i^{\vee}), \quad i = 1, \dots, n+1, \text{ and all monomials } a \text{ in } \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}],$$

$$\sum_{l=0}^s (\alpha_i^{\vee} \otimes a^{s-l}) p_{\alpha_i,a}^l, \quad \text{for } |s| > \Lambda(\alpha_i^{\vee}), \quad i = 1, \dots, n+1, \text{ and all monomials } a \text{ in } \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].$$

Let

$$\mathbf{A}_{\Lambda} = \mathcal{U}(\mathfrak{h}_{aff} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]) / I_{\Lambda}.$$

Using (4.5) it is easy to see that the algebra  $\mathbf{A}_{\Lambda}$  is generated by the elements of the set

$$\{\alpha_i^{\vee} \otimes t_2^{s_2} t_3^{s_3} \dots t_k^{s_k} : |s_l| \leq \Lambda(\alpha_i^{\vee}), \text{ for } 2 \leq l \leq k, \quad i = 2, \dots, n+1\}.$$

Hence  $\mathbf{A}_{\Lambda}$  is a finitely generated commutative algebra. Further it follows from (4.5) that given a weight vector  $v_0 \in V_{aff}^+ \cap V_{\Lambda}$ , the  $\mathfrak{h}_{aff} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ -submodule of  $V_{aff}^+$  generated by  $v_0$  is a quotient of the algebra  $\mathbf{A}_{\Lambda}$ . Thus the irreducible  $\mathfrak{h}_{aff} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ -submodules of  $V_{aff}^+$  generated by a non-zero vector of weight  $\Lambda \in P_{aff}^+$  are in one-to-one correspondence with the maximal ideals of  $\mathbf{A}_{\Lambda}$ , that is, the irreducible  $\mathfrak{h}_{aff} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ -submodules of  $V_{aff}^+$  generated by a non-zero vector of weight  $\Lambda \in P_{aff}^+$  are in one-to-one correspondence with the set of graded algebra homomorphisms from  $\mathbf{A}_{\Lambda}$  to  $\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ .

**4.3.** Set  $\mathcal{L}^c(\mathfrak{g}) := \mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathcal{Z}_1 \oplus \mathbf{C}d_1$ . Clearly  $\mathfrak{h}_{fin} \oplus \mathbf{C}K_1 \oplus \mathbf{C}d_1 = \mathfrak{h}_{aff}$  is the Cartan subalgebra of  $\mathcal{L}^c(\mathfrak{g})$ . Let  $\text{ev}(\mathbf{1}) : \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow \mathbf{C}$  be the evaluation map defined by  $\text{ev}(\mathbf{1})(t^{\mathbf{m}}) = 1$  for  $t^{\mathbf{m}} \in \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ .

Let  $V$  be an irreducible  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(m\mathbf{e}_1)}$  and  $v_0 \in V_{aff}^+$  be a non-zero vector of weight  $\Lambda$ . Then following our discussion in (4.2) there exists a graded algebra homomorphism  $\phi : \mathbf{A}_\Lambda \rightarrow \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  such that  $Y.v_0 = \phi(Y).v_0$ , for all  $Y \in \mathbf{A}_\Lambda$ . Given  $\phi : \mathbf{A}_\Lambda \rightarrow \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ , let  $\bar{\phi} : \mathbf{A}_\Lambda \rightarrow \mathbf{C}$  be the algebra homomorphism defined by the composition of functions  $\bar{\phi} := \text{ev}(\mathbf{1}) \circ \phi$  and let  $W_{\bar{\phi}} := \mathcal{U}(\mathcal{L}^c(\mathfrak{g})).v_0$  be the integrable  $\mathcal{L}^c(\mathfrak{g})$  module generated by the vector  $v_0$  such that  $Y.v_0 = \text{ev}(\mathbf{1}).\phi(Y)v_0 = \bar{\phi}(Y).v_0$ , for all  $Y \in \mathbf{A}_\Lambda$ . Since

$$\mathfrak{n}_{aff}^+ \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].v_0 = 0$$

and by definition the highest weight space  $\mathcal{U}(\mathfrak{h}_{fin} \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathcal{Z}_1).v_0$  of  $W_{\bar{\phi}}$  is one-dimensional,  $W_{\bar{\phi}}$  has a unique irreducible quotient  $V_{\bar{\phi}}$ .

**Lemma.** Let  $\Lambda \in P_{aff}^+$ . Given an algebra homomorphism  $\psi : \mathbf{A}_\Lambda \rightarrow \mathbf{C}$ , let  $W_\psi$  be the integrable  $\mathcal{L}^c(\mathfrak{g})$ -module generated by a vector  $v$  of weight  $\Lambda$  such that

$$\mathfrak{n}_{aff}^+ \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].v = 0, \quad Y.v = \psi(Y)v \quad \forall Y \in \mathbf{A}_\Lambda, \quad K_1.v = mv.$$

Then  $W_\psi$  is an integrable  $\mathcal{L}^c(\mathfrak{g})$ -module with finite-dimensional weight spaces.

*Proof.*  $W_\psi$  is an integrable  $\mathcal{L}^c(\mathfrak{g})$ -module and hence can be written as

$$W_\psi = \bigoplus_{\mu \in \mathfrak{h}_{aff}^*} W_\psi^\mu, \quad \text{where } W_\psi^\mu = \{u \in W_\psi : h.u = \mu(h)u, \forall h \in \mathfrak{h}_{aff}\}.$$

Let  $P(W_\psi) := \{\mu \in \mathfrak{h}_{aff}^* : W_\psi^\mu \neq 0\}$ . Since  $W_\psi$  is an integrable  $\mathcal{L}^c(\mathfrak{g})$ -module with highest weight  $\Lambda \in P_{aff}^+$ , an element  $\mu \in P(W_\psi)$  is of one of the following forms:

- (i).  $\mu = \Lambda - r\delta_1$ ,  $r \in \mathbf{Z}_{\geq 0}$ .
- (ii).  $\mu = \Lambda + \eta - r\delta_1$ ,  $\eta \in Q$  and  $r \in \mathbf{Z}_{\geq 0}$ .

We prove that in each of the cases  $W_\psi^\mu$  is spanned by a finite set of elements and hence is finite-dimensional.

**Case (i).** Let  $\mu = \Lambda - r\delta_1$  with  $r \in \mathbf{Z}_{\geq 0}$ . By definition  $\dim W_\psi^\Lambda = 1$ . Suppose now that  $r > 0$ . Then any element in  $W_\psi^{\Lambda - r\delta_1}$  is of the form  $(h_1 \otimes t_1^{-r_1} t^{\bar{\mathbf{r}}_1})(h_2 \otimes t_1^{-r_2} t^{\bar{\mathbf{r}}_2}) \cdots (h_s \otimes t_1^{-r_s} t^{\bar{\mathbf{r}}_s}).v$  with  $r_i \in \mathbf{Z}_{>0}$  such that  $\sum_{i=1}^s r_i = r$ ,  $t^{\bar{\mathbf{r}}_i} \in \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  and  $h_i \in \mathfrak{h}_{fin}$  for  $1 \leq i \leq s$ . Since  $\Delta_{fin}^\vee := \{\alpha_i^\vee : 1 \leq i \leq n\}$  is a basis of  $\mathfrak{h}_{fin}$  it follows that any element in  $W_\psi^{\Lambda - r\delta_1}$  is spanned by elements of the form

$$(\alpha_{i_1}^\vee \otimes t_1^{-r_1} t^{\bar{\mathbf{r}}_1})(\alpha_{i_2}^\vee \otimes t_1^{-r_2} t^{\bar{\mathbf{r}}_2}) \cdots (\alpha_{i_s}^\vee \otimes t_1^{-r_s} t^{\bar{\mathbf{r}}_s}).v, \quad (4.6)$$

with  $r_i \in \mathbf{Z}_{>0}$  such that  $\sum_{j=1}^s r_j = r$ ,  $t^{\bar{\mathbf{r}}_j} \in \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  and  $\alpha_{i_j} \in \Delta_{fin}^\vee$  for  $1 \leq j \leq s$ . Observe now that by putting  $a = t_1^{-r_1} t_2^{r_2} \cdots t_{i-1}^{r_{i-1}} t_{i+1}^{r_{i+1}} \cdots t_k^{r_k}$  and using Garland's equation (Lemma 2.6) we get

$$(x_{\alpha_j}^+ \otimes a^{-1} t_i)^{\Lambda(\alpha_j^\vee)} (x_{\alpha_j}^- \otimes a)^{\Lambda(\alpha_j^\vee)+1}.v = \sum_{j=0}^{\Lambda(\alpha_j^\vee)} x_{\alpha_j}^- \otimes a(t_i)^{\Lambda(\alpha_j^\vee)-j} p_{\alpha_j, t_i}^j.v = 0. \quad (4.7)$$

Applying  $x_{\alpha_j}^+ \otimes t_i^p$  to (4.7) we get

$$-\alpha_j^\vee \otimes a(t_i)^{\Lambda(\alpha_j^\vee)+p} = \sum_{\ell=1}^{\Lambda(\alpha_j^\vee)} \alpha_j^\vee \otimes a(t_i)^{\Lambda(\alpha_j^\vee)-\ell+p} p_{\alpha_j, t_i}^\ell.v, \quad \forall p \in \mathbf{Z}. \quad (4.8)$$

As  $K_i t^{\mathbf{r}}.v = 0$  for all  $1 \leq i \leq k$  whenever  $\mathbf{r} \in \mathbf{Z}^k$  is such that  $r_1 \neq 0$  it follows from (4.8) and the fact that  $W_\psi^{\Lambda-r\delta_1}$  is spanned by elements of the form (4.6), that  $W_\psi^{\Lambda-r\delta_1}, r \in \mathbf{Z}_{>0}$  lies in that span of elements in the set

$$S_{\Lambda-r\delta_1} = \{(\alpha_{i_1}^\vee \otimes t_1^{-r_1} t^{\bar{\mathbf{r}}_1})(\alpha_{i_2} \otimes t_1^{-r_2} t^{\bar{\mathbf{r}}_2}) \cdots (\alpha_{i_s} \otimes t_1^{r_s} t^{\bar{\mathbf{r}}_s}).v : \alpha_{i_j} \in \Delta_{fin}^\vee, r_j \in \mathbf{Z}_{>0}, r = \sum_{j=1}^s r_j, \\ \bar{\mathbf{r}}_j = (0, r_2^j, \dots, r_k^j) \text{ with } |r_t^j| < \Lambda(\alpha_{i_j}^\vee) \text{ } 1 \leq j \leq s, 2 \leq t \leq k\}.$$

Since  $\Delta_{fin}^\vee$  is finite and the number of partitions of a given positive number is finite it follows the length of any element in  $W_\psi^{\Lambda-r\delta_1}$  is less than  $r$ . Hence the set  $S_{\Lambda-r\delta_1}$  is finite implying that  $W_\psi^{\Lambda-r\delta_1}$  is finite-dimensional.

**Case (ii.).** Let  $\mu = \Lambda + \eta - r\delta_1$  with  $r \in \mathbf{Z}_{>0}$  and  $\eta \in Q$ . Note that the elements in  $W_\psi^{\Lambda+\eta-r\delta_1}$  are of the form

$$(Y_1 \otimes t_1^{-r_1} t^{\bar{\mathbf{r}}_1})(Y_2 \otimes t_2^{-r_2} t^{\bar{\mathbf{r}}_2}) \cdots (Y_s \otimes t_1^{-r_s} t^{\bar{\mathbf{r}}_s}).v, \quad (4.9)$$

with  $Y_j \in \mathfrak{g}_{fin}$  for  $1 \leq j \leq s$ ,  $r_j < 0$  whenever  $Y_j \in \mathfrak{n}_{fin}^+ \oplus \mathfrak{h}_{fin}$  and  $r_j \leq 0$  for  $Y_j \in \mathfrak{n}_{fin}^-$ . We prove by induction on  $s$  that any such element is in the span of the elements

$$(Y_1 \otimes t^{\mathbf{m}_1}) \cdots (Y_\ell \otimes t^{\mathbf{m}_\ell}).v$$

where for all  $1 \leq t \leq \ell$ ,  $Y_t \in \mathfrak{g}_{fin}$  and  $\mathbf{m}_t = (m_1^t, m_2^t, \dots, m_k^t) \in \mathbf{Z}^k$  is such that  $m_1^t \leq 0$  and for  $2 \leq j \leq k$ ,  $|m_j^t| < \max_{1 \leq i \leq n} \Lambda(\alpha_i^\vee)$  if  $Y_t \in \mathfrak{h}_{fin}$  and  $|m_j^t| < \Lambda(\gamma_{r_t}^\vee)$  if  $Y_t \otimes t_1^{-r_t}$  is a real root vector of  $\mathfrak{g}_{aff}$  of weight  $\gamma - r_t \delta_1$ .

Suppose  $s = 1$ . Then  $\eta \in R_{fin}$ ,  $r_1 = r$  and  $Y_1 \otimes t_1^{-r} \in \mathfrak{n}_{aff}^-$  is of the form  $x_\eta \otimes t_1^{-r}$ . Hence there exists  $y_\eta \otimes t_1^r \in \mathfrak{n}_{aff}^+$  such that the triple  $\mathfrak{sl}_2(\eta, -r) = \{y_\eta \otimes t_1^r, x_\eta \otimes t_1^{-r}, \eta_r^\vee := [y_\eta \otimes t_1^r, x_\eta \otimes t_1^{-r}]\}$  is isomorphic to  $\mathfrak{sl}_2(\mathbf{C})$ . Setting  $a = t_2^{r_2} \cdots t_{i-1}^{r_{i-1}} t_{i+1}^{r_{i+1}} \cdots t_k^{r_k}$  and using Garland's equation we get

$$(y_\eta \otimes t_1^r a^{-1} t_i)(x_\eta \otimes t_1^{-r} a)^{\Lambda(\eta_r^\vee)+1}.v = \sum_{j=0}^{\Lambda(\eta_r^\vee)} x_\eta \otimes t_1^{-r} a t_i^{\Lambda(\eta_r^\vee)-j} p_{\eta+r\delta_1, t_i}^j.v$$

Applying  $\eta^\vee \otimes t_i^p$  to the above equation we see that for any real root  $\gamma \in R_{aff}^+$ , the element  $x_\gamma^- \otimes t^{\bar{\mathbf{r}}}.v$ , with  $\bar{\mathbf{r}} = (0, r_2, \dots, r_k)$ , lies in the span of  $x_\gamma^- \otimes t_2^{s_2} \cdots t_k^{s_k}$  where  $-\Lambda(\gamma^\vee) < s_j < \Lambda(\gamma^\vee)$ ,  $2 \leq j \leq k$ . If an element in  $W_\psi^{\Lambda+\eta-r\delta_1}$  is of the form  $(Y_1 \otimes t_1^{-r_1} t^{\bar{\mathbf{r}}_1})(Y_2 \otimes t_1^{r_2} t^{\bar{\mathbf{r}}_2}).v$  with  $Y_1 \in \mathfrak{n}_{fin}^+ \oplus \mathfrak{n}_{fin}^-$  and  $Y_2 \in \mathfrak{h}_{fin}$ , then using the Lie bracket operation in  $\mathcal{L}^c(\mathfrak{g})$ , the relations (4.8) and induction step  $s = 1$  it follows such elements lie in the linear span of elements of the form

$$(Y_1 \otimes t_1^{-r_1} t^{\bar{\mathbf{m}}_1})(Y_2 \otimes t_1^{-r_2} t^{\bar{\mathbf{m}}_2}).v, \quad (4.10)$$

where for  $2 \leq j \leq k$ ,  $|m_j^2| < \max_{1 \leq i \leq n} \Lambda(\alpha_i^\vee)$  and  $|m_j^1| < \Lambda(\eta_{r_1}^\vee)$ . Since the number of pairs  $(r_1, r_2) \in \mathbf{Z}_{\geq 0}$

such that  $r_1 + r_2 = r$  is finite, it follows that every element in  $W_\psi^{\Lambda+\eta-r\delta_1}$  of the form (4.10) lies in the span of a finite set of elements. Suppose now that the result is true when  $1 \leq s < t$  and consider an element in  $W_\psi^{\Lambda+\eta-r\delta_1}$  of length  $t$ . Clearly

$$(Y_1 \otimes t_1^{-r_1} t^{\bar{\mathbf{r}}_1})(Y_2 \otimes t_1^{-r_2} t^{\bar{\mathbf{r}}_2}) \cdots (Y_s \otimes t_1^{-r_s} t^{\bar{\mathbf{r}}_s}).v = (Y_2 \otimes t_1^{-r_2} t^{\bar{\mathbf{r}}_2})(Y_1 \otimes t_1^{-r_1} t^{\bar{\mathbf{r}}_1}) \cdots (Y_s \otimes t_1^{-r_s} t^{\bar{\mathbf{r}}_s}).v \\ + [Y_1 \otimes t_1^{-r_1} t^{\bar{\mathbf{r}}_1}, Y_2 \otimes t_1^{-r_2} t^{\bar{\mathbf{r}}_2}](Y_3 \otimes t_1^{-r_3} t^{\bar{\mathbf{r}}_3}) \cdots (Y_s \otimes t_1^{-r_s} t^{\bar{\mathbf{r}}_s}).v.$$

Since  $[Y_1 \otimes t_1^{-r_1} t^{\bar{\mathbf{r}}_1}, Y_2 \otimes t_1^{-r_2} t^{\bar{\mathbf{r}}_2}] \in \mathfrak{g}_{fin} \otimes t_1^{-r_1-r_2} t^{\bar{\mathbf{r}}_1+\bar{\mathbf{r}}_2}$ , applying an induction argument on the length  $t$  identical to the one used in [CP3, Proposition 1.2(ii)] we see that elements of the form (4.9) are spanned by elements of the desired form. Since  $R_{fin}$  is finite, for a fixed  $\eta \in Q$  there can exist only finitely many tuples  $(\beta_1, \beta_2, \dots, \beta_t) \in (R_{fin})^t$  such that  $\sum_{j=1}^t \beta_j = \eta$ , further the number of partitions of a

given positive number is finite. Hence given  $\eta \in Q$  and  $r \in \mathbb{N}$  there exists a positive integer  $N_{\eta,r}$  such that every element in  $W_{\psi}^{\Lambda+\eta-r\delta_1}$  is of length less than equal to  $N_{\eta,r}$ . This shows that the set

$$\begin{aligned} S_{\Lambda+\eta-r\delta_1} = \{ & (Y_1 \otimes t^{\mathbf{m}_1}) \cdots (Y_{\ell} \otimes t^{\mathbf{m}_{\ell}}).v : Y_j \in \mathfrak{g}_{fin} \text{ with } \sum_{j=1}^{\ell} \text{wt}_{fin}(Y_j) = \eta, r_j \geq 0 \text{ with } \sum_{j=1}^{\ell} r_j = r, \\ & \bar{\mathbf{r}}_j = (0, r_2^j, \dots, r_k^j) \text{ with } |r_p^j| < \max_{1 \leq i \leq n} \Lambda(\alpha_i^{\vee}) \text{ if } Y_j \in \mathfrak{h}_{fin}, \\ & |r_p^j| \leq \Lambda(\gamma_{r_j}^{\vee}) \text{ if } \text{wt}_{aff}(Y_j \otimes t_1^{-r_j}) = \gamma - r_j \delta_1 \ \forall 2 \leq p \leq k\}. \end{aligned}$$

is finite and since every element in  $W_{\psi}^{\Lambda+\eta-r\delta_1}$  lies in the span of elements from  $S_{\Lambda+\eta-r\delta_1}$  it follows that  $W_{\psi}^{\Lambda+\eta-r\delta_1}$  is finite dimensional for  $\eta \in Q$  and  $r \geq 0$ .  $\square$

We thus conclude that given an irreducible  $\mathcal{T}(\mathfrak{g})$ -module  $V$  one can associate with it an unique irreducible  $\mathcal{L}^c(\mathfrak{g})$ -module having finite-dimensional weight spaces. This leads us to the study of the irreducible integrable representations of  $\mathcal{L}^c(\mathfrak{g})$  having finite-dimensional weight spaces.

**4.4.** The following result leads towards the classification of the irreducible integrable  $\mathcal{L}^c(\mathfrak{g})$ -modules having finite-dimensional weight spaces.

**Proposition.** *Let  $\mathcal{V}$  be an integrable irreducible representation of  $\mathcal{L}^c(\mathfrak{g})$  with finite-dimensional weight spaces. Suppose  $\mathcal{V}$  is generated by a vector  $\mathbf{v}$  such that*

$$\mathfrak{n}_{aff}^+ \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].\mathbf{v} = 0, \quad h.\mathbf{v} = \Lambda(h)\mathbf{v}, \quad \forall h \in \mathfrak{h}_{aff}, \quad K_1.\mathbf{v} = m\mathbf{v},$$

for  $\Lambda \in P_{aff}^+$ . Then  $\mathcal{V}$  is isomorphic to an irreducible module for a finite direct sum of affine Kac Moody Lie algebras  $\mathfrak{g}_{aff}$ .

*Proof.* Let  $\psi : \mathcal{L}^c(\mathfrak{g}) \rightarrow \text{End } \mathcal{V}$  be an irreducible integrable representation of  $\mathcal{L}^c(\mathfrak{g})$  having finite-dimensional weight spaces such that  $\psi(K_1) = m \text{id}_{\bar{\mathcal{V}}}$  for  $m \in \mathbf{Z}_{>0}$ . Clearly such a module is  $d_1$ -graded, therefore  $\ker \psi$  is an ideal of  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathbb{Z}_1$ . By definition  $\mathbb{Z}_1$  is in the center of  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathbb{Z}_1$ . Hence the adjoint representation of  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathbb{Z}_1$  can be identified with a representation of  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ . This implies that any ideal of  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathbb{Z}_1$  (which is a module of  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathbb{Z}_1$  with respect to the adjoint representation) is an ideal of the Lie algebra  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ . In particular  $\ker \psi$  is an ideal of  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  and by Lemma 2.3 there exists an ideal  $S$  of  $\mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  such that  $\ker \psi = \mathfrak{g}_{fin} \otimes S$ . If  $\mathbb{Z}_1^S$  is the central extension of the Lie algebra  $\mathfrak{g}_{fin} \otimes (\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}]/S)$ , then it follows that the irreducible  $\mathcal{L}^c(\mathfrak{g})$ -module  $\mathcal{V}$  is a faithful representation of the Lie algebra  $\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathbb{Z}_1^S \oplus \mathbb{C}d_1$ . Hence the action of the subalgebra  $\mathfrak{h}_{aff} \otimes (\mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}])$  of  $\mathcal{U}(\mathcal{L}^c(\mathfrak{g}))$  on  $\mathbf{v}$  is given by the restriction of the map  $\psi$  to the Lie algebra  $\mathfrak{h}_{aff} \otimes (\mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/S)$ .

On the other hand since  $\mathcal{V}$  is irreducible, it follows from 4.2 that the  $\mathfrak{h}_{aff} \otimes \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ -module generated by  $\mathbf{v}$  corresponds to a maximal ideal of the commutative algebra  $\mathbf{A}_{\Lambda}$ , or equivalently to an algebra homomorphism  $\phi : \mathbf{A}_{\Lambda} \rightarrow \mathbb{C}$ . Thus for any  $h \in \mathfrak{h}_{aff}$  and  $a \in \mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  we have,

$$h \otimes a.\mathbf{v} = \phi(h \otimes a).\mathbf{v} = \psi(h \otimes a).\mathbf{v} = h \otimes \bar{a}.v_0,$$

where  $\bar{a}$  is the image of  $a$  in  $\mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/S$ . Since  $\phi(h \otimes a) \in \mathbb{C}$ , we conclude that the ideal  $S$  is equal to the intersection of distinct maximal ideals  $\{\mathcal{M}_j\}_{j \in J}$  of  $\mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ . By Chinese remainder theorem it thus follows that an irreducible integrable  $\mathcal{L}^c(\mathfrak{g})$ -module with finite-dimensional weight spaces is in fact an irreducible representation of a Lie algebra of the form

$$(\oplus_{j \in J} (\mathfrak{g}_{fin} \otimes \mathbb{C}[t_1^{\pm 1}] \oplus \mathbb{C}K_1) \otimes (\mathbb{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/\mathcal{M}_j)) \oplus \mathbb{C}d_1$$



whenever it is generated by a highest weight vector of weight  $\Lambda \in P_{aff}^+$  with  $\Lambda(K_1) \neq 0$ .

It is well known that an irreducible representation of  $\bigoplus_{j \in J} (\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}] \oplus \mathbf{C}K_1) \otimes (\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/\mathcal{M}_j) \oplus \mathbf{C}d_1$  is a tensor product of irreducible representations of the Lie algebras

$$\{(\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}] \oplus \mathbf{C}K_1) \otimes (\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/\mathcal{M}_j) \oplus \mathbf{C}d_1 : j \in J\}.$$

Hence for each  $j \in J$  there exists an irreducible representation  $V_j$  of  $(\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}] \oplus \mathbf{C}K_1) \otimes (\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/\mathcal{M}_j) \oplus \mathbf{C}d_1$  such that  $\mathcal{V} = \bigoplus_{j \in J} V_j$ . As a consequence the highest weight vector  $v_0 \in V$  is of the form

$$\mathbf{v} = \bigotimes_{j \in J} v_j,$$

where  $v_j$  is a weight vector of  $V_j$  for  $j \in J$ . Since the maximal ideals are distinct, it is easy to see that

$$\mathbf{n}_{aff}^+ \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].\mathbf{v} = 0, \quad \text{if and only if} \quad \mathbf{n}_{aff}^+ \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}].v_j = 0, \quad \forall j \in J.$$

Hence for each  $j \in J$ , the vector  $v_j \in V_j$  is of weight  $\mu_j$  where  $\mu_j \in P_{aff}^+$  is such that  $\mu_j(K_1) > 0$  and for all  $h \in \mathfrak{h}_{aff}$ ,

$$h.\mathbf{v} = \Lambda(h)(v) = \left( \sum_{j \in J} \mu_j(h) \right) \bigotimes_{j \in J} v_j.$$

In particular,  $\sum_{j \in J} \mu_j(\alpha_{n+1}^\vee) = m < \infty$ . Hence the set  $J$  is finite which completes the proof of the proposition.  $\square$

It follows directly from Proposition 4.4 that given an algebra homomorphism  $\phi : \mathbf{A}_\Lambda \rightarrow \mathbf{C}$  one can uniquely associate with it a finitely supported function from  $\max \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow P_{aff}^+$  that maps a maximal ideal  $\mathcal{M}_j$  of  $\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  to the weight of the highest weight vector of the irreducible  $(\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}] \oplus \mathbf{C}K_1) \otimes (\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/\mathcal{M}_j) \oplus \mathbf{C}d_1$  module under  $\phi$ .

**4.5.** Given  $M \in \max \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  let  $\text{ev}_M : \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow \mathbf{C}$  be the evaluation map at the point in  $(\mathbf{C}^*)^{k-1}$  corresponding to the maximal ideal  $M$  of  $\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ .

Let  $\Pi$  be the monoid of finitely supported functions  $\pi : \max \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow P_{aff}^+$ . For  $\pi, \pi' \in \Pi$  and  $M \in \max \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  define

$$(\pi + \pi')(M) = \pi(M) + \pi'(M), \quad \text{supp}(\pi) = \{M \in \max \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] : \pi(M) \neq 0\},$$

$$\text{wt}(\pi) = \sum_{M \in \text{supp}(\pi)} \pi(M).$$

For  $\pi \in \Pi$ , let  $M_1, M_2, \dots, M_l$  is an enumeration of  $\text{supp}(\pi)$  and let  $X_\pi = \bigotimes_{i=1}^l X(\pi(M_i))$ , be the  $\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathcal{Z} \oplus \mathbf{C}d_1$ -module in which the action of the Lie algebra is defined as follows:

$$Y \otimes f.v_1 \otimes \dots \otimes v_l = \sum_{i=1}^l \text{ev}_{M_i}(f)v_1 \otimes \dots \otimes Y.v_i \otimes \dots \otimes v_l,$$

where  $Y \in \mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}] \oplus \mathbf{C}K_1 \oplus \mathbf{C}d_1$  and  $f \in \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$ . Let

$$L(X_\pi) := X_\pi \otimes \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}],$$

be the  $\mathcal{T}(\mathfrak{g})$ -module on which the action of  $\mathcal{T}(\mathfrak{g})$  is given as follows:

$$\begin{aligned} Y \otimes f.(w \otimes f') &= (Y \otimes f.w) \otimes f f', & K_j t^{\mathbf{m}}.(w \otimes f') &= 0, \quad \forall 2 \leq j \leq k, \mathbf{m} \in \mathbf{Z}^k, \\ d_i.(w \otimes f') &= w \otimes d_i(f'), \quad \text{for } 2 \leq i \leq k, & d_1.w \otimes f' &= d_1(w) \otimes f', \end{aligned}$$

for  $Y \in \mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}] \oplus \mathbf{C}K_1 \oplus \mathbf{C}d_1$ ,  $f, f' \in \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  and  $w \in X_\pi$ . For  $M \in \text{supp}(\pi)$ , let  $v_M$  be the highest weight vector of  $X(\pi(M))$  and let  $v_\pi := \otimes_{i=1}^l v_{M_i}$ . Suppose  $\text{wt}(\pi) = \Lambda$ . Then with respect to the action of  $\mathcal{T}(\mathfrak{g})$  on  $L(X_\pi)$ , it is clear that the  $\mathbf{A}_\Lambda$ -module generated by  $v_\pi$  is  $\mathbf{Z}^{k-1}$ -graded. Hence setting

$$A_\Lambda^\pi := \mathbf{A}_\Lambda \cdot v_\pi,$$

we can write

$$A_\Lambda^\pi = \bigoplus_{\mathbf{m} \in \mathbf{Z}^{k-1}} A_\Lambda^\pi[\mathbf{m}].$$

Let

$$G_\pi := \{\mathbf{m} = (m_2, \dots, m_k) \in \mathbf{Z}^{k-1} : A_\Lambda^\pi[\mathbf{m}] \neq 0\}.$$

**Lemma.** *Let  $\Lambda \in P_{aff}^+$  and  $\pi \in \Pi$  be such that  $\text{wt}(\pi) = \Lambda$ . Then the set  $G_\pi$  is a subgroup of  $\mathbf{Z}^{k-1}$  of rank  $k-1$ .*

*Proof.* Given  $\pi \in \Pi$  with  $\text{wt}(\pi) = \Lambda \in P_{aff}^+$ , there exists some  $h \in \mathfrak{h}_{aff}$  such that  $h \cdot v_\pi \neq 0$ . Hence  $\mathbf{0} \in G_\pi$ . As the  $\mathbf{A}_\Lambda$ -module  $A_\Lambda^\pi$  generated by  $v_\pi$  corresponds to an algebra homomorphism on  $\mathbf{A}_\Lambda$ , the set  $G_\pi$  is closed under addition. By 3.4 every algebra homomorphism from  $\mathbf{A}_\Lambda$  to  $\mathbf{C}$  corresponds to an irreducible representation of  $\mathbf{A}_\Lambda$ . Hence  $A_\Lambda^\pi$  is an irreducible  $\mathbf{A}_\Lambda$ -module and consequently every  $\mathbf{Z}^{k-1}$ -graded element of  $A_\Lambda^\pi$  is invertible. That is if  $A_\Lambda^\pi[\mathbf{m}] \neq 0$ , then  $A_\Lambda^\pi[-\mathbf{m}] \neq 0$ . This implies that  $G_\pi$  is closed under inverses and hence  $G_\pi$  is a subgroup of  $\mathbf{Z}^{k-1}$ .

Suppose  $M_1, M_2, \dots, M_l$  is an enumeration of  $\text{supp}(\pi)$  then for any  $h \in \mathfrak{h}_{aff}$  and  $2 \leq i \leq k$ , we have

$$h \otimes t_i^r \cdot v_\pi = \left( \sum_{i=1}^l \pi(M_i)(h) \text{ev}_{M_i}(t_i^r) \right) v_\pi \otimes t_i^r.$$

Since  $\text{supp}(\pi)$  is finite and  $\pi(M_i)(h) \in \mathbf{Z}_+$  for all  $M_i \in \text{supp}(\pi)$  and  $h \in \mathfrak{h}_{aff}$ ,  $(\sum_{i=1}^l \pi(M_i)(h) \text{ev}_{M_i}(t_i^r))$  cannot be zero for all  $h \in \mathfrak{h}_{aff}$  and  $r \in \mathbf{Z}_+$ . Hence for all  $\pi \in \Pi$ , rank of  $G_\pi$  is  $k-1$ , implying that  $G_\pi$  is a subgroup of  $\mathbf{Z}^{k-1}$  of finite index.  $\square$

Given  $\pi \in \Pi$ , we shall henceforth refer to the set  $G_\pi$  as the group associated to  $\pi$ .

Set  $\mathbf{C}[t^{G_\pi}]$  as the set of all polynomials in the variables  $\{t^{\mathbf{m}} : \mathbf{m} \in G_\pi\}$ . As  $G_\pi$  is a subgroup of  $\mathbf{Z}^{k-1}$  of rank  $k-1$ , it is easy to see that  $\mathbf{C}[t^{G_\pi}]$  is isomorphic to a Laurent polynomial ring in  $k-1$  variables. Let  $G^\pi = \mathbf{Z}^{k-1}/G_\pi$ . Clearly  $G^\pi$  is a finite group. For  $\mathbf{g} \in G^\pi$ , let  $\mathbf{C}t^{\mathbf{g}}[t^{G_\pi}]$  be the set of all polynomials in the variables  $\{t^{\mathbf{g}+\mathbf{m}} : \mathbf{m} \in G_\pi\}$ . From the construction of  $G_\pi$  it follows that the irreducible  $\mathbf{A}_\Lambda$ -module  $A_\Lambda^\pi$  is isomorphic to  $v_\pi \otimes \mathbf{C}[t^{G_\pi}]$  and for each  $\mathbf{g} \in G^\pi$ , the irreducible  $\mathbf{A}_\Lambda$ -module generated by the vector  $v_\pi \otimes t^{\mathbf{g}} \in L(X_\pi)$  is isomorphic to the subspace  $v_\pi \otimes \mathbf{C}t^{\mathbf{g}}[t^{G_\pi}]$  of  $L(X_\pi)$ .

The following result was proved in [R3, Proposition 3.5, Theorem 3.18, Example 4.2].

**Proposition.** *For  $\pi \in \Pi$ , let  $v_\pi$  be the highest weight vector of  $X_\pi$  and let  $G_\pi$  be the group associated to  $\pi$  and  $G^\pi = \mathbf{Z}^{k-1}/G_\pi$ . Then we have the following.*

i. *For each  $\mathbf{g} \in G^\pi$ , the  $\mathcal{T}(\mathfrak{g})$ -module*

$$X_\pi^{\mathbf{g}} = \mathcal{U}(\mathcal{T}(\mathfrak{g})) \cdot v_\pi \otimes t^{\mathbf{g}},$$

*is an irreducible  $\mathcal{T}(\mathfrak{g})$ -module.*

ii.  *$L(X_\pi)$  is completely reducible as a  $\mathcal{T}(\mathfrak{g})$ -module. In fact as a  $\mathcal{T}(\mathfrak{g})$ -module  $L(X_\pi)$  is isomorphic to the direct sum of the irreducible  $\mathcal{T}(\mathfrak{g})$ -modules  $X_\pi^{\mathbf{g}}$ ,  $\mathbf{g} \in G^\pi$ , that is,*

$$L(X_\pi) \cong_{\mathcal{T}(\mathfrak{g})} \bigoplus_{\mathbf{g} \in G^\pi} X_\pi^{\mathbf{g}}.$$

Further if  $V$  is an irreducible  $\mathcal{T}(\mathfrak{g})$ -module in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$ , then upto twisting by one-dimensional  $\mathcal{T}(\mathfrak{g})$ -modules  $V$  is isomorphic to  $X_\pi^{\mathfrak{g}}$  for some  $\pi \in \Pi$  with  $\text{wt}(\pi)(\alpha_{n+1}) = m$  and  $\mathfrak{g} \in G_\pi$ .

**4.6.** Notice that for  $\mathbf{b} = (b_2, \dots, b_k) \in (\mathbf{C}^*)^{k-1}$  the map  $s_{\mathbf{b}} : \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  given by  $t_i \mapsto b_i t_i$ , for  $i = 2, \dots, k$  is an isomorphism. Using this isomorphism we now give a parametrization of the irreducible representations of  $\mathcal{T}(\mathfrak{g})$  in  $\mathcal{I}_{fin}^{(me_1)}$ ,  $m > 0$ . In the case when the center acts trivially, the isomorphism classes of irreducible  $\mathcal{T}(\mathfrak{g})$ -modules in  $\mathcal{I}_{fin}$  was determined in [CFK, PB, L].

**Theorem.** Given  $\pi, \pi' \in \Pi$ ,  $\mathfrak{g} \in \mathbf{Z}^{k-1}/G_\pi$  and  $\mathfrak{g}' \in \mathbf{Z}^{k-1}/G_{\pi'}$ , the irreducible  $\mathcal{T}(\mathfrak{g})$ -module  $X_\pi^{\mathfrak{g}}$  is isomorphic to  $X_{\pi'}^{\mathfrak{g}'}$  if and only if there exists  $\mathbf{b} = (b_2, \dots, b_k) \in (\mathbf{C}^*)^{k-1}$  such that

- (i)  $\text{supp}(\pi') = \{s_{\mathbf{b}}(M) : M \in \text{supp}(\pi)\}$ ,
- (ii) For all  $M \in \text{supp}(\pi)$ ,  $X(\pi(M))$  is isomorphic to  $X(\pi'(s_{\mathbf{b}}(M)))$  as a  $\mathfrak{g} \otimes \mathbf{C}[t_1, t_1^{-1}] \oplus \mathbf{C}K_1$ -module.
- (iii)  $\mathfrak{g} \equiv \mathfrak{g}' \pmod{G_\pi}$ .

*Proof.* Suppose  $\chi : X_\pi^{\mathfrak{g}} \rightarrow X_{\pi'}^{\mathfrak{g}'}$  is a  $\mathcal{T}(\mathfrak{g})$ -module isomorphism from  $X_\pi^{\mathfrak{g}}$  to  $X_{\pi'}^{\mathfrak{g}'}$ . If  $M_1, M_2, \dots, M_r$  is an enumeration of  $\text{supp}(\pi)$ , then by Proposition 4.4  $X_\pi^{\mathfrak{g}}$  is a module for the Lie algebra

$$\bigoplus_{i=1}^r ((\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}]) \otimes (\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/M_i)) \oplus \mathcal{Z} \oplus D_k.$$

Hence via the isomorphism  $\chi$ ,  $X_{\pi'}^{\mathfrak{g}'}$  is a module for the Lie algebra  $\bigoplus_{i=1}^r ((\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}]) \otimes (\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/M_i)) \oplus \mathcal{Z} \oplus D_k$ . By definition however  $X_{\pi'}^{\mathfrak{g}'}$  is a module for the Lie algebra

$$\bigoplus_{M \in \text{supp}(\pi')} ((\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}]) \otimes (\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]/M)) \oplus \mathcal{Z} \oplus D_k.$$

Hence there exists an isomorphism  $\sigma : \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  such that for  $1 \leq i \leq r$ ,  $\sigma(M_i) = \mathcal{M}_j^i$  for  $\mathcal{M}_j^i \in \text{supp}(\pi')$ , or equivalently there exists  $\mathbf{b} = (b_2, \dots, b_k) \in (\mathbf{C}^\times)^{k-1}$ , such that

$$\sigma = s_{\mathbf{b}} : \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}] \rightarrow \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$$

is the isomorphism of  $\mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  given by  $t_i \mapsto b_i t_i$  for  $2 \leq i \leq k$ . This implies  $\{s_{\mathbf{b}}(M) : M \in \text{supp}(\pi)\} \subseteq \text{supp}(\pi')$ . Since  $\chi$  and  $s_{\mathbf{b}}$  are isomorphisms, taking the inverse maps we see that

$$\text{supp}(\pi') = \{s_{\mathbf{b}}(M) : M \in \text{supp}(\pi)\}.$$

Let  $f_1, \dots, f_r \in \mathbf{C}[t_2^{\pm 1}, \dots, t_k^{\pm 1}]$  be such that  $f_i \in \bigcap_{j=1, j \neq i}^r M_j$ . Then for each  $x \in \mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}] \oplus \mathbf{C}K_1 \oplus \mathbf{C}d_1$  we have,

$$\chi(x \otimes f_i \cdot v_\pi \otimes t^{\mathfrak{g}}) = x \otimes s_{\mathbf{b}}(f_i) \cdot \chi(v_\pi \otimes t^{\mathfrak{g}}) \quad \text{for } 1 \leq i \leq r.$$

Using the definition of the action of  $\mathcal{T}(\mathfrak{g})$  on  $L(X_\pi)$  it follows that if  $\chi$  is an isomorphism, then the  $\mathfrak{g}_{aff}$  module generated by  $v_{M_i}$  is isomorphic to the  $\mathfrak{g}_{aff}$ -module generated by  $v_{s_{\mathbf{b}}(M_i)}$ . Hence by [VV, Theorem 3, Lemma 1] for every  $M \in \text{supp}(\pi)$ ,  $X(\pi(M))$  is isomorphic to  $X(\pi'(s_{\mathbf{b}}(M)))$  as a  $\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1, t_1^{-1}] \oplus \mathbf{C}K_1$ -module.

As a consequence of conditions (i) and (ii),  $G_\pi = G_{\pi'}$  whenever  $X_\pi^{\mathfrak{g}}$  is isomorphic to  $X_{\pi'}^{\mathfrak{g}'}$  implying that  $\mathfrak{g}, \mathfrak{g}' \in \mathbf{Z}^{k-1}/G_\pi$ . Since the set of highest weight vectors in  $X_{\pi'}^{\mathfrak{g}'}$  are contained in the subspace  $v'_\pi \otimes t^{\mathfrak{g}'} \mathbf{C}[t^{G'_\pi}] = v'_\pi \otimes t^{\mathfrak{g}'} \mathbf{C}[t^{G_\pi}]$ , and the highest weight vector  $v_\pi \otimes t^{\mathfrak{g}}$  of  $X_\pi^{\mathfrak{g}}$  maps to a highest weight vector in  $X_{\pi'}^{\mathfrak{g}'}$  we have,  $d_i(\chi(v_\pi \otimes t^{\mathfrak{g}})) = \chi(d_i \cdot v_\pi \otimes t^{\mathfrak{g}}) = g_i \chi(v_\pi \otimes t^{\mathfrak{g}})$ ,  $\forall 2 \leq i \leq k$ . This is possible if and only if  $\mathfrak{g} \equiv \mathfrak{g}' \pmod{G_\pi}$ . This completes the proof of the theorem.  $\square$

Using the same proof as above it is easy to see the following.

**Proposition.** *Given  $\pi, \pi' \in \Pi$ , the irreducible  $\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}] \oplus \mathcal{Z}$ -module  $X_\pi$  is isomorphic to  $X_{\pi'}$  if and only if there exists  $\mathbf{b} = (b_2, \dots, b_k) \in (\mathbf{C}^*)^{k-1}$  such that*

- (i)  $\text{supp}(\pi') = \{s_{\mathbf{b}}(M) : M \in \text{supp}(\pi)\},$
- (ii) *For  $M \in \text{supp}(\pi)$ ,  $X(\pi(M))$  is isomorphic to  $X(\pi'(\mathbf{b}.M))$  as a  $\mathfrak{g}_{fin} \otimes \mathbf{C}[t_1, t_1^{-1}] \oplus \mathbf{C}K_1$ -modules.*

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